CHAPTER 3

Sequences

1. Limits and the Archimedean Property

Our first basic object for investigating real numbers is the sequence. Before we give the precise definition of a sequence, we will give the intuitive description. To begin, a **finite sequence** is just a finite ordered list of real numbers. For example, $(1, 2, 3, 4, 5)$ is a sequence with five terms and $(\pi, e, 3, 3)$ is a sequence with four.

We added the adjective 'ordered' above to reflect the fact that the order of the terms matters. For example, the 2 term sequence $(\pi, 7)$ is not the same as the 2 term sequence $(7, \pi)$. Notice that a sequence is not the same as a set for which order (and also repetition) do not matter. The sets $\{7, \pi\}$ and $\{\pi, 7\}$ are the same. Likewise the sets $\{3\}$ and $\{3, 3\}$ are the same, whereas the sequences (3) and $(3, 3)$ are not.

So far we have been discussing finite sequences, but for this course, these objects will not be the focus of our study. For us, a sequence will always be an infinite sequence. As you have probably guessed, an infinite sequence is essentially an infinite ordered list of real numbers. We now give the precise definition.

Definition. A **sequence** is a function $f : \mathbb{N} \to \mathbb{R}$.

At this point you decide for yourself whether the definition we have given captures the intuitive idea we were seeking. Giving a precise definition for an intuitive idea is a very important skill in studying mathematics.

For example, the functions defined by $f(n) = n^2$ and $g(n) = \frac{1}{n}$ are both sequences.

Although a sequence is technically a function, we typically do not use functional notation to discuss sequences as this notation does not reflect the intuitive notion we are attempting to describe. Instead of writing something like $f(n) = n^2$ or $g(n) = \frac{1}{n}$ to denote a sequence, we write $(n^2)_{n=1}^{\infty}$ or $(\frac{1}{n})_{n=1}^{\infty}$. If we say "consider the sequence $(a_n)_{n=1}^{\infty}$," we are referring to the sequence whose value at n is the real number a_n (in other words the sequence whose nth term is a_n).

Like any function, a sequence does not have to be defined via an elementary formula: any random list of numbers will work. For example, a sequence could be defined by saying the nth term is the nth decimal of π (we have not actually defined decimal expansions yet, but this infinite list of numbers certainly does not follow an elementary formula).

One good way to define a sequence is **recursively**: one states the first term (or the first several terms) and then gives a rule for getting each term from the previous ones.

3.1. A famous example of a recursively defined sequence is the **Fibonacci sequence**. It is defined by

$$a_1 = 1, \qquad a_2 = 1, \qquad a_{n+2} = a_{n+1} + a_n \text{ for } n \geq 1.$$

Find the third, fourth, and fifth terms for the Fibonacci sequence.

Although it is usually defined recursively, it is actually possible to give a formula for the Fibonacci sequence. This need not be the case for a recursively defined sequence.

A **constant sequence** is a sequence whose every term is the same number. For example, $(1, 1, 1, \dots,)$ is the constant sequence of value one. Once again we emphasize that a sequence is not to be confused with a set. For example, the set $\{1, 1, 1, \dots\}$ is really just the set $\{1\}$ but $(1, 1, 1, \dots)$ denotes the function $f : \mathbb{N} \to \mathbb{R}$ with $f(n) = 1$ for all n. It is true that every sequence gives a set, namely the set of values that it takes (in other words the range of the corresponding function), but different sequences can give the same set (and you should give some examples of this occurrence).

Before we continue, we should point a common convention of the notation here. Sometimes we will write "consider the sequence $(a_n)_{n=4}^{\infty}$." To be precise, this means our sequence is given by the function $f(n) = a_{3+n}$ for $n \in \mathbb{N}$. We do this because it is often notationally convenient to do something like "starting the sequence at $n = 4$."

We now proceed to discuss the notion of convergence. We cannot overstate the importance of this concept. In fact, it is easily the most importance concept of this chapter and in understanding the structure of the real numbers (beyond the algebraic and ordering properties we described in the appendices). In some sense the entire purpose of introducing sequences is to give a framework under which we can study convergence in $\mathbb{R}$.

All that being said, many students find the definition of convergence a bit confusing (at least at first) and so you should certainly attempt to think carefully about it and from several different points of view (and we will attempt to help you do so). As with sequences, we will begin by giving an intuitive description.

Intuitively, a sequence converges to a number L if the terms of the sequence are 'heading towards L' as we go down the list. In one attempt to formulate a definition, one might say that a sequence $(a_n)_{n=1}^{\infty}$ converges to a limit L if "the terms of (a_n) get closer and closer to L as n gets larger and larger." For example the sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$ is clearly getting closer and closer to zero as we proceed through the list.

However, though they shed some light on the concept, the words we have said above do not quite capture the idea fully. For instance, we might also consider the sequence $(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots)$. This sequence is certainly also heading to zero and yet the terms are not always getting closer to zero as we go along. For example, the first occurrence of 0 is certainly closer to 0 than is the later term $\frac{1}{2}$. Furthermore, returning to our previous example of $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$, we see that terms of the sequence are also in fact getting closer and closer to -1 (or any other number less than zero).

Thus is perhaps more appropriate to say that $(a_n)_{n=1}^{\infty}$ converges to L if "the terms of (a_n) get arbitrarily close to L as n gets larger." This definition solves the problems to which we have already alluded, but it introduces some other linguistic difficulties. Specifically, it may not be clear exactly what we mean by the phrase 'arbitrarily close'. To be a little clearer, we might say that no matter how close we want to be to the limit L, there is a point in the sequence past which we are always at least that close to L. The language here is more specific, but it also more convoluted. Expressing precise mathematical ideas in plain language is often quite difficult, but the attempt can be a very beneficial exercise.

We now give the precise definition.

Definition. A sequence $(a_n)_{n=1}^{\infty}$ is said to **converge** to $L \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n \geq N$, $|a_n - L| < \epsilon$.

Again this definition might appear a bit confusing and so you should think about it carefully. Does this match the intuitive idea we had in mind? How are we measuring closeness to L? What is the purpose of the number N? Can you choose different values of N for two different values of ϵ? What is the relationship between ϵ and N? Come up with some other open-ended questions to ask yourself.

Although the intuitive idea of convergence has been around for quite some time, this precise definition seems to have been first published by Bernard Bolzano, a Czech mathematician, in 1816. As you might have gleaned from your calculus courses, it is the notion of convergence or of limits that distinguishes analysis/calculus from, say, algebra. Nevertheless, the precise formulation came about 150 years after the creation of calculus (due independently to Newton and Leibniz). The fact that it will probably take you some time to understand and become comfortable with it is therefore no surprise: it took even the world's most brilliant mathematicians more than a century to nail it down precisely (of course they were trying to accomplish the task without the aid of textbooks and instructors).

Definition. If the sequence $(a_n)_{n=1}^{\infty}$ converges to L, L is called a **limit** of the sequence. If there exists any $L \in \mathbb{R}$ such that $(a_n)_{n=1}^{\infty}$ converges to L then we say $(a_n)_{n=1}^{\infty}$ **converges** or that $(a_n)_{n=1}^{\infty}$ is a **convergent** sequence.

So far we have not actually shown that a number sequence cannot have more than one limit. If we want our intuitive understanding of limit to be satisfied, we will certainly want this to be the case. You will now show that it is, beginning with the following helpful lemma.

3.2. Let $a \geq 0$ be a real number. Prove that if for every $\epsilon > 0$ we have $a < \epsilon$ then $a = 0$.

3.3. Prove that if $(a_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ and $(a_n)_{n=1}^{\infty}$ converges to $M \in \mathbb{R}$ then $L = M$.

Thus it makes sense to talk about *the* limit of a sequence (rather than *a* limit of a sequence). If the limit of $(a_n)_{n=1}^{\infty}$ is L we often write $\lim\limits_{n\to\infty} a_n = L$. In other words, $\lim\limits_{n\to\infty} a_n$ is the limit of the sequence $(a_n)_{n=1}^{\infty}$. Similarly, we often write $a_n \to L$ if $\lim_{n\to\infty} a_n = L$.

Caution: Before we write $\lim\limits_{n\to\infty} a_n$ we must know that a_n has a limit: we will see below that many sequences do not have limits.

We have emphasized repeatedly that one of the keys to understanding a definition is creating and understanding examples. At this point you should come up with some sequences for which you can figure out the limit. We have already mentioned that $(1/n)_{n=1}^{\infty}$ should converge to 0. Can you give some other examples? The intuition that you have developed in you calculus courses should be helpful.

Of course there is a difference between knowing intuitively that something is true (or being told it is true by an authority figure like a teacher) and having a mathematical proof (and thus a rigorous understanding) of the fact. We now need to move from the former to the latter, beginning with the simplest sequences.

3.4. Show using the definition of convergence that the constant sequence of value $a \in \mathbb{R}$ converges to a.

3.5. Prove using the definition of a limit that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Let's examine the proof here. Your proof should look something like this.

PROOF. We will show $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Let $\epsilon > 0$ be arbitrary but fixed. We must find $N \in \mathbb{N}$ so that if $n \geq N$ then $|\frac{1}{n} - 0| < \epsilon$ which is the same as $\frac{1}{n} < \epsilon$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Then if $n \geq N$, $\frac{1}{n} \leq \frac{1}{N} < \epsilon$. □

The proof above relies on two things. Firstly, we used the basic properties of order which we have assumed are known. Secondly, we have used the fact that, given an $\epsilon > 0$, there exists $N \in \mathbb{N}$ with $\frac{1}{N} < \epsilon$ or in other words with $\frac{1}{\epsilon} < N$. This is actually a fact that needs to be proven, but we will temporarily take it as known.

Explicitly, we assume that $\mathbb{N} \subset \mathbb{R}$ has the **Archimedean Property**, which says that for every $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ with $x < n$. The proof of this seemingly obvious fact is surprisingly delicate. In fact, it necessarily relies on the completeness axiom which we have yet to formulate. This is an example of a situation where a seemingly obvious fact is not so obvious when we attempt to prove it.

3.6. Prove the following consequence to the Archimedean property: For every $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < \epsilon.$$

Armed with the Archimedean property and its consequences, we can now study some more concrete examples of limits precisely.

3.7. Prove using the definition of a limit that

$$\lim_{n\to\infty} 1 - \frac{1}{n^2+1} = 1 \text{ and } \lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}.$$

3.8. Let $(a_n)_{n=1}^{\infty}$ be sequence defined by saying that $a_n = \frac{1}{n}$ for n odd and $a_n = 0$ for n even. Show that $\lim_{n\to\infty} a_n = 0$.

We will see in this chapter that convergent sequences have many special and important properties. We give the first now.

Definition. A sequence $(a_n)_{n=1}^{\infty}$ is called **bounded** if the associated set $\{a_n : n \in \mathbb{N}\}$ is contained in a bounded interval.

Notice that (a_n) is bounded if and only if there exists $K \geq 0$ with $|a_n| \leq K$ for all $n \in \mathbb{N}$. (why?)

3.9. Prove that if $(a_n)_{n=1}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} a_n = L$ then $(a_n)_{n=1}^{\infty}$ is bounded.

Now that we have a considered a few affirmative examples, we need to consider some negatives ones. In other words, we need to think about what it means for a sequence *not* to converge.

3.10. Negate the definition of $\lim_{n\to\infty} a_n = L$ to give an explicit definition of "$(a_n)_{n=1}^{\infty}$ does not converge to L."

We can write $(a_n)_{n=1}^{\infty}$ does not converge to L as $(a_n)_{n=1}^{\infty} \not\to L$.

Caution: We should not write $\lim_{n\to\infty} a_n \neq L$ to mean $(a_n)_{n=1}^{\infty}$ does not converge to L unless we know that $\lim_{n\to\infty} a_n$ exists.

Definition. A sequence (a_n) is said to **diverge** or be **divergent** if it does not converge to L for any $L \in \mathbb{R}$.

3.11. Without making a reference to the definition of convergence, formulate in precise logical language (as in the definition of convergence) a definition for '$(a_n)_{n=1}^{\infty}$ is divergent.' Avoid using a phrase like 'there exists no L.' Compare your definition to the one you gave for '$(a_n)_{n=1}^{\infty}$ does not converge to L'.

Definition. We will distinguish two special types of divergence:

(1) A sequence (a_n) is said to **diverge to** $+\infty$ if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \geq M$. In an abuse of notation we often write $\lim_{n\to\infty} a_n = +\infty$ or $a_n \to +\infty$.
(2) A sequence (a_n) is said to **diverge to** $-\infty$ if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \leq M$. Again in an abuse of notation we often write $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$.

In fact, we need to justify this terminology.

3.12. Show that if (a_n) diverges to ∞ then a_n diverges. Likewise, show that if (a_n) diverges to $-\infty$, it diverges.

Thus one way for a sequence to diverge is for it to 'head off to $\pm\infty$' (i.e., to ∞ or $-\infty$). This is, however, not the only way.

3.13. We have seen that every convergent sequence is bounded. Give an example of a sequence which is bounded and yet divergent. Show that it does not diverge to $\pm\infty$.

3.14. Which, if any, of the following conditions are equivalent to $(a_n)_{n=1}^{\infty}$ converges to L. If a condition is equivalent, prove it. If not, give a counter-example.

(1) There exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n \geq N$ with $|a_n - L| < \epsilon$.
(2) For all $\epsilon > 0$ and for all $N \in \mathbb{N}$, there is an $n \geq N$ with $|a_n - L| < \epsilon$.
(3) For all $N \in \mathbb{N}$, there exists an $\epsilon > 0$ such that for all $n \geq N$, $|a_n - L| < \epsilon$.
(4) For all $N \in \mathbb{N}$ and $n \geq N$, there is an $\epsilon > 0$ with $|a_n - L| < \epsilon$.

2. Properties of Convergence

Now that we have established the basic terminology of convergence (and of divergence), we need to study all the basic properties. These properties will both help us to understand limits and help us to prove things about limits.

3.15. Prove that a sequence $(a_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ if and only if for all $\epsilon > 0$ the set

$$\{n \in \mathbb{N} : |a_n - L| \geq \epsilon\}$$

is finite.

Thus a sequence converges to L if and only if, for each $\epsilon > 0$, the number of terms which are more than ϵ distance from L is finite. This observation has a couple of interesting corollaries.

3.16. Prove:

(1) If we change finitely many terms of a sequence, we do not alter its limiting behavior: if the sequence originally converged to L then the altered sequence still converges to L, and if the original sequence diverged to $\pm\infty$ or diverged in general then so does the altered sequence.
(2) If we remove a finite number of terms from a sequence then we do not alter its limiting behavior.

The previous results confirm a fact which is perhaps intuitively clear: if we change or remove some terms at the beginning of a sequence, we do not change where it is headed.

3.17. Let $S \subset \mathbb{R}$ be a set. Assume that for all $\epsilon > 0$ there is an $a \in S$ with $|a| < \epsilon$. Prove there there is a convergent sequence $(a_n)_{n=1}^{\infty}$, with $a_n \in S$ for all n, and $\lim_{n\to\infty} a_n = 0$.

3.18. Prove or disprove:

(1) If $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} |a_n| = |L|$.
(2) If $\lim_{n\to\infty} |a_n| = |L|$ then $\lim_{n\to\infty} a_n = L$.
(3) If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.
(4) If $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} |a_n| = 0$.

This next result is often called the **Squeeze Theorem** for sequences.

3.19. If $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ then $\lim_{n\to\infty} c_n = L$.

We next prove a collection of results known as the **Limit Laws** for sequences. In fact, this name is a misleading as essentially all the results of this chapter could equally be given the same name (as they are all facts about limits). For historical reasons, the results given this name are those that describe the relation between limits and the algebraic properties of $\mathbb{R}$.

The following result is not a limit law, but it will be very useful in proving them (specifically the one related to division).

3.20. Let $(b_n)_{n=1}^{\infty}$ be a convergent sequence whose limit M is nonzero. Prove that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|b_n| > \frac{|M|}{2}$.

Suppose now that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences and $c \in \mathbb{R}$ is a real number. We can define new sequences $(c \cdot a_n)_{n=1}^{\infty}$, $(a_n + b_n)_{n=1}^{\infty}$, and $(a_n \cdot b_n)_{n=1}^{\infty}$. If $b_n \neq 0$ for all $n \in \mathbb{N}$ then we can define

$$\left(\frac{a_n}{b_n}\right)_{n=1}^{\infty}.$$

3.21. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences and let $c \in \mathbb{R}$. Prove the Limit Laws:

(1) If $(a_n)_{n=1}^{\infty}$ is a convergent sequence and $c \in \mathbb{R}$ then $(c \cdot a_n)_{n=1}^{\infty}$ is a convergent sequence and
$$\lim_{n\to\infty}(c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n.$$

(2) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences then $(a_n + b_n)_{n=1}^{\infty}$ is a convergent sequence and
$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(3) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences then $(a_n \cdot b_n)_{n=1}^{\infty}$ is a convergent sequence and
$$\lim_{n\to\infty}(a_n \cdot b_n) = \big(\lim_{n\to\infty} a_n\big) \cdot \big(\lim_{n\to\infty} b_n\big).$$

(4) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences with $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n \neq 0$ then
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

Hint: For the third part, the problem is that we have two quantities changing simultaneously. To deal with this we use a very common trick in analysis: we add and subtract additional terms, which does not affect the value, and then group terms so that each term is a product of things we can control. Let $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$. We can write

$$\begin{aligned} a_n \cdot b_n - L \cdot M &= a_n \cdot b_n - L \cdot M + L \cdot b_n - L \cdot b_n \\ &= (a_n - L) \cdot b_n + L \cdot (b_n - M). \end{aligned}$$

Hint: For the fourth part, given the third part, it suffices to prove (explain why) that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}.$$

Let $M = \lim_{n\to\infty} b_n$ and notice that

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|Mb_n|} < \frac{|M - b_n|}{(M^2/2)}$$

if $|b_n| > \frac{|M|}{2}$.

3.22. Give an example of two sequences (a_n) and (b_n) such that (a_n) and (b_n) diverge and yet $(a_n + b_n)$ converges. Give an example where (a_n) and (b_n) diverge and yet $(a_n b_n)$ converges.

This next result is also sometimes including among the Limit Laws. Needless to say, it gives the interaction between the notion of a limit and the ordering properties of $\mathbb{R}$.

3.23. Suppose $a \leq a_n \leq b$ for all $n \in \mathbb{N}$. Prove that if $\lim_{n\to\infty} a_n = L$, then $L \in [a, b]$. Prove that the conclusion is still true if a_n is outside $[a, b]$ for only finitely many n.

3.24. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences with $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Prove that if $a_n - b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ then $a = b$. Would this theorem still be true if, instead of equality, you had $a_n - b_n < \frac{1}{n}$? What if $a_n - b_n = \frac{1}{2^n}$?

3. Monotone Sequences

In this section, we will finally formulate the completeness axiom. To do so, we introduce another class of sequences.

Definition. Let (a_n) be a sequence. We say that (a_n) is **increasing** if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$. Likewise, we say that (a_n) is **decreasing** if for all $n \in \mathbb{N}$, $a_n \geq a_{n+1}$. A sequence is called **monotone** if it is either increasing or decreasing.

3.25. Prove the following:

(1) If (a_n) is increasing and unbounded then (a_n) diverges to $+\infty$.
(2) If (a_n) is decreasing and unbounded then (a_n) diverges to $-\infty$

Thus we see that an increasing sequence can diverge if it "escapes to $+\infty$." Suppose though that we decided ahead of time that this was not allowed. In other words consider an increasing sequence which is also bounded above. For the sake of intuition, assume that the sequence is **strictly increasing** meaning that each term is strictly larger than the last (i.e., not equal to the last).

In some sense we should expect that this sequence should be 'trapped.' On the one hand, the terms of the sequence are getting larger and larger. On the other, we have assumed that there is a ceiling that they cannot break. Of course, there is no guarantee that they will every get very close to our ceiling. But that just means we could pick a smaller ceiling. If they don't get close to that one either, we can just pick another and so on. In this way, we should be able to trap the sequence into smaller and smaller spaces (as the terms gets larger).

Thus it seems reasonable that these terms should be 'headed' somewhere specific. However, if you try to make a rigorous proof out of our intuitive ideas, it will always fail. Thus to make our idea work, we have to add an assumption to our axioms for the real numbers. This is the essence of the *completeness axiom.* The **completeness axiom** states that every bounded increasing sequence of real numbers converges.

In the introduction to chapter 2, we said that the completeness axiom forbids the existence of 'gaps' in the real line. In some sense, our formulation of the completeness axiom captures this idea because it asserts that there are no gaps towards which the 'trapped' sequence can head: it has to be heading towards a specific number. Our use of the word 'gap' here is an attempt to express an abstract mathematical ideal in plain language (a theme we have emphasized throughout the course). Nevertheless, we will see below that it is not a perfect choice of words. Perhaps after you complete and understand the material of this chapter, you can give a better explanation (and you should certainly try).

We can however finally prove the Archimedean property.

3.26. Suppose that $x \in \mathbb{R}$. Then there is some $n \in \mathbb{N}$ with $n > x$.

Hint: Consider the sequence $(n)_{n=1}^{\infty}$. If the Archimedean property is false then it is bounded. Since it is increasing, it must have limit L. Consider precisely the implications of assuming that all the natural numbers head towards a fixed number L.

You will notice that you used the completeness axiom in your proof. This is not an accident: it is impossible to prove the Archimedean property without assuming the completeness axiom . More precisely, the Archimedean property cannot be proven using the other axioms alone. We can view this fact as yet more motivation for assuming the completeness axiom (as it seems like a strange thing to assume at first). Certainly the set $\mathbb{N}$ should not be bounded by a real number.

3.27. Suppose that (a_n) is decreasing and bounded. Than (a_n) converges.

We will now give several results which demonstrate the power of the completeness axiom. We begin by studying an important class of sequences known as **geometric sequences**.

3.28. Let $r \in \mathbb{R}$.

(1) Prove that if $0 \leq r < 1$ then $(r^n)_{n=1}^{\infty}$ converges to 0.
(2) Prove that if $r > 1$ then $(r^n)_{n=1}^{\infty}$ diverges to $+\infty$.
(3) Prove that the sequence (r^n) converges if and only if $-1 < r \leq 1$. If $r = 1$ prove that $\lim_{n\to\infty} r^n = 1$. If $-1 < r < 1$, prove that $\lim_{n\to\infty} r^n = 0$.

Hint: For the first part, show the sequence is decreasing and bounded below. Let L be the limit and proceed by contradiction: suppose $L > 0$, find a_n with $L \leq a_n < r^{-1}L$, and hence produce a contradiction.

3.29. Let $a_1 = 2$ and let (a_n) be generated by the recursive formula

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

for $n \geq 1$.

(1) Prove that a_n is well-defined and positive.
(2) Prove that $2 < a_n^2$ for all $n \in \mathbb{N}$.
(3) Prove that a_n is decreasing and hence converges.

(4) Now take limits on both sides of the recursive formula and prove and use the fact that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

to show that $a^2 = 2$ if $a = \lim_{n\to\infty} a_n$. Show that $a > 0$.

Hint: For the second part, assuming $a_n > 0$ turn

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) > \sqrt{2}$$

into an equivalent condition on a quadratic polynomial. Proceed by induction.

Needless to say the real number a considered in the previous result is usually denoted by $\sqrt{2}$. Thus the completeness axiom also implies that $\sqrt{2}$ is a well-defined real number.

3.30. $\sqrt{2}$ is not a rational number.

Of course we call a real number which is not rational an **irrational number**. Thus we have proven the ('obvious') fact that there exist irrational numbers. In other words, the collection of real numbers is larger than the collection of rational numbers. We will, however, see now that it is not too much larger (in an appropriate sense).

3.31. Prove the following using the Archimedean property:

(1) for every $x \in \mathbb{R}$ there exists an $m \in \mathbb{Z}$ such that

$$m \leq x < m + 1,$$

(2) for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists an $m \in \mathbb{Z}$ such that

$$\frac{m}{n} \leq x < \frac{m+1}{n},$$

Using these we can prove that every real number can be approximated arbitrarily well by a rational number. The following result is referred to by saying that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

3.32. Prove:

(1) For every $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\frac{m}{n} \in \mathbb{Q}$ such that

$$0 \leq x - \frac{m}{n} < \epsilon.$$

(2) For every $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\frac{m}{n} \in \mathbb{Q}$ such that

$$-\epsilon < x - \frac{m}{n} \leq 0$$

(3) For every $x \in \mathbb{R}$ and $\epsilon > 0$ there exists an $a \in \mathbb{Q}$ with $|x - a| < \epsilon$.

(4) For every $x \in \mathbb{R}$ there is a sequence (x_n) of rational numbers with $\lim_{n\to\infty} x_n = x$.

The previous result shows a weakness of the language we used in our attempt to describe the completeness axiom. We stated that the truth of the completeness axiom was tantamount to saying that $\mathbb{R}$ contains no gaps. But this result implies that if we draw a 'gap' (i.e., an interval) on the real line, no matter how small, there is always a rational number inside of it. Thus there are no gaps of this kind in $\mathbb{Q}$ either, despite the fact that $\mathbb{Q}$ is not complete (i.e., does not satisfy the completeness axiom: there are bounded increasing sequence in $\mathbb{Q}$ which do not converge to a limit in $\mathbb{Q}$). It might be better to say that the completeness axiom states that $\mathbb{R}$ has no 'infinitely-small gaps.' But perhaps at this point our language starts to lose meaning.

The irrational numbers are also dense in $\mathbb{R}$.

3.33. Prove:

(1) For all $\frac{m}{n} \in \mathbb{Q} \setminus \{0\}$ the number $\sqrt{2}\ \frac{m}{n}$ is irrational.

(2) For every $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\frac{m}{n} \in \mathbb{Q}$ such that

$$0 \leq x - \sqrt{2}\frac{m}{n} < \epsilon.$$

(3) For every $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\frac{m}{n} \in \mathbb{Q}$ such that

$$-\epsilon < x - \sqrt{2}\frac{m}{n} \leq 0$$

3.34. Let $x \in \mathbb{R}^+$. What can you say about the sequence given by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{x}{a_n})$ for $n \geq 1$?

3.35. For each n, let $I_n = [a_n, b_n]$ be a bounded closed interval. Suppose that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

(1) Prove that there exists a $p \in \mathbb{R}$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

(2) Suppose in addition that $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. Prove that if $q \in I_n$ for all $n \in \mathbb{N}$ then $q = p$.

4. Subsequences

Intuitively, a subsequence of the sequence (a_n) is a (new) sequence formed by skipping (possibly infinitely many) terms in a_n.

Definition. A sequence $(b_k)_{k=1}^{\infty}$ is a **subsequence** of $(a_n)_{n=1}^{\infty}$ if there exists a strictly increasing sequence of natural numbers $n_1 < n_2 < \cdots$ so that for all $k \in \mathbb{N}$, $b_k = a_{n_k}$.

Again you should see for yourself that this definition captures the intuitive idea above. As an example, $(1, 1, 1, \ldots)$ is a subsequence of $(1, -1, 1, -1, \ldots)$. In fact, any sequence of ± 1's is a subsequence of $(1, -1, 1, -1, \ldots)$.

In particular, we see that a subsequence of a divergent sequence may be convergent.

3.36. Give an example of a sequence $(a_n)_{n=1}^{\infty}$ of natural numbers (i.e., with $a_n \in \mathbb{N}$ for each n) so that every sequence of natural numbers is a subsequence of (a_n). Can you do the same if $\mathbb{N}$ is replaced with $\mathbb{Z}$?

$(\frac{1}{n^2})_{n=1}^{\infty}$ is a subsequence of $(\frac{1}{n})_{n=1}^{\infty}$, and both $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ and $\left(\frac{1}{n^2}\right)_{n=1}^{\infty}$ converge to 0. The next problem asks if this example can be generalized.

3.37. Prove or disprove: If (b_n) is a subsequence of (a_n) and $\lim\limits_{n\to\infty} a_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Subsequences are useful because, if a specified sequence does not have a certain desirable property, we can often find a subsequence which does.

3.38. Prove that every sequence of real numbers has a monotone subsequence.

Hint: Consider two cases. The first is the case in which every subsequence has a minimum element.

Thus we get the following very useful result.

3.39. Prove that every bounded sequence of real numbers has a convergent subsequence.

5. Cauchy Sequences

One of the reasons that the completeness axiom is so strong is that it (by definition) allows us to conclude that certain sequences converge without knowing their limit beforehand (for example we were able to show that $\sqrt{2}$ exists as a real number without assuming it beforehand).

But bounded monotone sequences are certainly not the only type of sequence which converges. In this section, we give a condition which is equivalent to convergence, but makes no reference to knowing the limit of the sequence.

Definition. A sequence (a_n) is called **Cauchy** if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $|a_n - a_m| < \epsilon$.

3.40. Negate the definition of Cauchy sequence to give an explicit definition of "$(a_n)_{n=1}^{\infty}$ is not a Cauchy sequence."

3.41. Which of the following conditions, if any, are equivalent to "$(a_n)_{n=1}^{\infty}$ is Cauchy" or "(a_n) is not Cauchy?"

(1) For all $\epsilon > 0$ and $N \in \mathbb{N}$ there are $n, m > N$ with $n \neq m$ and $|a_n - a_m| < \epsilon$.
(2) For all $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that for all $p, q \geq N$, $|a_p - a_q| < 1/n$.
(3) There exists an $\epsilon > 0$ and $n \neq m$ in $\mathbb{N}$ with $|a_n - a_m| < \epsilon$.
(4) There exists $N \in \mathbb{N}$ such that for all $\epsilon > 0$ there are $n \neq m$ with $n, m > N$ and $|a_n - a_m| < \epsilon$.

3.42. Prove that every convergent sequence is Cauchy.

This is particularly useful in the contrapositive form: if $(a_n)_{n=1}^{\infty}$ is not Cauchy then $(a_n)_{n=1}^{\infty}$ diverges.

3.43. Prove that every Cauchy sequence is bounded. Is the converse true?

3.44. Let (a_n) be a Cauchy sequence and let (a_{n_k}) be a convergent subsequence. Prove that (a_n) is convergent and $\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{n_k}$.

We are thus led to the following extremely important result.

3.45. Show that a sequence is convergent if and only if it is Cauchy.

Hence, a Cauchy sequence in $\mathbb{R}$ is the same as a convergent sequence in $\mathbb{R}$. Again the advantage is that the definition of a Cauchy sequence makes no reference to the limit: it is an intrinsic property of the sequence.

3.46. Is every Cauchy sequence in $\mathbb{Q}$ convergent to some point in $\mathbb{Q}$?

We remark that the equivalence of Cauchy sequence and convergent sequence is itself actually equivalent to the completeness axiom. In other words, one can assume that Cauchy sequences converge and prove that bounded monotone sequences converge. Many books begin with the Cauchy perspective and call the above equivalence the completeness axiom (the fact that bounded monotone sequence converge is then called the 'monotone convergence theorem').

3.47. Prove or give a counterexample: if a sequence of real numbers $(x_n)_{n=1}^{\infty}$ has the property that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_{n+1} - x_n| < \epsilon$, then (x_n) is a convergent sequence. How is this different from the definition of a Cauchy sequence?

6. Decimals

So far we have not had a systematic way of describing real numbers. In other words, unless we are dealing with a special number like a rational number or $\sqrt{2}$, we have no systematic way to specify a number. We will attempt to solve this problem by using decimal expansions.

Of course this method will fall somewhat short as many decimal expansions cannot be written down or even specified easily. In some sense, the only decimal expansion that we can easily write down either terminate

$$\frac{1}{8} = 0.125 \qquad \frac{27}{50} = 0.54$$

or become periodic

$$\frac{1}{3} = 0.3333\ldots = 0.\overline{3} \qquad \frac{1}{7} = 0.142857142857\ldots = 0.\overline{142857}$$

and we will see that such expansions do not encompass all numbers.

3.48. Write down a decimal expansion that is not periodic in such a way that the pattern is clear.

It is an interesting fact that a decimal expansion which either terminates or becomes periodic always represents a rational number (and we will show this fact below).

Another 'problem' with decimal expansions is that a single number can have two different expansions. For example, we will see that

$$\frac{1}{8} = 0.124\overline{9} = 0.125.$$

Fortunately, we will also see that all the numbers that have multiple decimal expansions are in fact rational. However, not all rational numbers have multiple expansions.

Shortly we will prove that every real number has a decimal expansion. For example the real number π has a decimal expansion even though not all the digits are known. As of this writing, the first 2,699,999,990,000 decimal digits have been calculated. Ironically a Frenchman named Fabrice Bellard was able to complete this calculation on his home computer using a new algorithm of his own design. It took Bellard's computer 131 days to complete the program. This accomplish is particulary impressice since the previous record was calculated on a Japanese super computer known as the T2K Open Supercomputer.

Chao Lu of China holds the Guinness Book of Records record for reciting digits of π. In just over twenty-four hours, he recited the first $67,890$ digits of π. He was given no breaks of any kind as the rules stated that he had to recite each digit within 15 seconds of the previous one.

We now define and construct decimal expansions precisely. Consider the interval $[0, 1]$. Firstly, we divide the interval $[0, 1]$ into 10 equal closed subintervals (of length $\frac{1}{10}$):

$$\left[0, \tfrac{1}{10}\right] \cup \left[\tfrac{1}{10}, \tfrac{2}{10}\right] \cup \left[\tfrac{2}{10}, \tfrac{3}{10}\right] \cup \left[\tfrac{3}{10}, \tfrac{4}{10}\right] \cup \left[\tfrac{4}{10}, \tfrac{5}{10}\right] \cup \left[\tfrac{5}{10}, \tfrac{6}{10}\right] \cup \cdots \cup \left[\tfrac{9}{10}, 1\right].$$

Label these intervals $I_0, I_1, \ldots I_9$, respectively.

Now let k_1 be an integer between 0 and 9. Then I_{k_1} is the interval $I_{k_1} = [\frac{k_1}{10}, \frac{k_1+1}{10}]$. We may further divided this interval into ten equal pieces (of length $\frac{1}{100}$):

$$\left[\frac{k_1}{10}, \frac{k_1}{10} + \frac{1}{100}\right] \cup \left[\frac{k_1}{10} + \frac{1}{100}, \frac{k_1}{10} + \frac{2}{100}\right] \cup \cdots \cup \left[\frac{k_1}{10} + \frac{9}{100}, \frac{k_1+1}{10}\right].$$

Label these intervals $I_{k_1,0}, I_{k_1,2}, \ldots, I_{k_1,9}$. By dividing further and further, we may define the interval $I_{k_1,k_2,\ldots,k_n}$ where $(k_1, \ldots, k_n)$ is any finite sequence with values in $0, \ldots, 9$.

3.49. What are the endpoints of the interval $I_{k_1,\ldots,k_n}$?

Definition. Suppose that $a \in [0,1]$. A **decimal expansion** for x is an infinite sequence, $(k_i)_{i=1}^{\infty}$, such that each $0 \le k_i \le 9$ is an integer and so that $a \in I_{k_1,\ldots,k_n}$ for each n.

This definition might seem a little bit strange at first. How does it compare to the picture you have in mind for a decimal expansion?

3.50. Explain how a number in $[0,1]$ can have more than 1 decimal expansion.

3.51. Suppose $a \in [0,1]$. First prove that we may find a decimal expansion for a. Next assume that $(k_i)_{i=1}^{\infty}$ is a decimal expansion for $a \in [0,1]$. Define a sequence $(a_n)_{n=1}^{\infty}$ by saying that

$$a_n = \frac{k_1}{10} + \frac{k_2}{100} + \cdots + \frac{k_n}{10^n} = \sum_{i=1}^{n} \frac{k_i}{10^i}.$$

Prove that (a_n) is convergent and $\lim_{n\to\infty} a_n = a$.

You may recall from your calculus course that we denote the limit of of the above sequence (a_n) by

$$\sum_{i=1}^{\infty} \frac{k_i}{10^i}.$$

There is a converse to the last result which perhaps demonstrates that our definition of decimal expansion is a reasonable one.

3.52. Suppose that $(k_i)_{i=1}^{\infty}$ is such that k_i is an integer with $0 \le k_i \le 9$. Show that

$$\sum_{i=1}^{\infty} \frac{k_i}{10^i}$$

always exists. Furthermore show that if

$$a = \sum_{i=1}^{\infty} \frac{k_i}{10^i}$$

then (k_i) is a decimal expansion for a.

It perhaps goes without saying that if (k_i) is a decimal expansion for a, we often write $a = 0.k_1k_2k_3\ldots$. Of course this notation does not reflect any kind of multiplication of the k_i. If we write a terminating decimal sequence like $0.k_1k_2\cdots k_n$, we mean the first n terms are $k_1, \ldots, k_n$ and all the other terms are zero.

So far we have only considered decimal expansions for numbers in $[0, 1]$. Of course we should define decimal expansions for all real numbers.

3.53. Extend the definition of decimal expansion to all real numbers. Explicitly state what it means for $(k_i)_{i=0}^{\infty}$ to be a decimal expansion for $a \in \mathbb{R}$. Show that the previous two results work for all real numbers and not just numbers in $[0, 1]$.

Again we usually denote the decimal expansion $(k_i)_{i=0}^{\infty}$ by $k_0.k_1k_2k_3\ldots$. Now that we have defined decimal expansions, we prove that all the basic results that we expect are true.

3.54. Suppose $x, y \in \mathbb{R}$ are not equal and suppose (k_i) and (ℓ_i) are decimal expansions for x and y respectively. In addition, assume without loss of generality that neither (k_i) nor (ℓ_i) ends with a constant sequence of 9's. Show that $x < y$ if and only if there exists a $r \in \mathbb{N}$ with $k_0.k_1k_2\ldots k_r < \ell_0.\ell_1\ell_2\ldots\ell_r$.

3.55. Suppose $x, y \in \mathbb{R}$ and suppose $\{k_i\}$ and $\{\ell_i\}$ are decimal expansions for x and y respectively. Describe (and prove) how to find a decimal expansion for $x + y$, for $-y$ and for $x - y$.

3.56. Suppose $x, y \in \mathbb{R}$ and suppose $\{k_i\}$ and $\{\ell_i\}$ are decimal expansions for x and y respectively. Describe (and prove) how to find a decimal expansion for xy, for $1/y$ and for x/y.

3.57. Prove that a decimal expansion is eventually periodic if and only if it comes from a rational number.

We conclude our study of decimal expansions by remarking that the ideas of this section can be used to construct the real numbers rigorously (rather than taking their existence on faith). One defines the real numbers to be the set of integer sequences $(k_i)_{i=0}^{\infty}$ such that $0 \le k_i \le 9$ for $i \ge 1$. We have to a little bit careful in that we need to specify that decimals that should give the same number are equal (this is accomplished by putting a equivalence relation on the set).

We then define addition, subtraction, multiplication, and division in the way that we did in the previous exercises. Likewise we define an order on the collection. We then verify that all the axioms (including the completeness axiom) hold. If you feel particularly ambitious, you could attempt to carry out this construction precisely and verify all the axioms. You certainly have the intellectual tools at your disposal, but the construction and verification might be a bit long and it will certainly take persistence.

7. Supremums and Completeness

In this section, we give yet another equivalent formulation of the completeness axiom. This version is a little bit different than the others in that it makes no mention of sequences and instead deals with sets. We begin with a few definitions.

Definition. Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

(1) x is called an **upper bound for** A if, for all $a \in A$, we have $a \leq x$.
(2) x is called an **lower bound for** A if, for all $a \in A$, we have $x \leq a$.
(3) The set A is called **bounded above** if there exists an upper bound for A. The set A is called **bounded below** if there exists a lower bound for A. The set A is called **bounded** if it is both bounded above and bounded below.
(4) x is called a **maximum of** A if $x \in A$ and x is an upper bound for A. We will often write $x = \max A$.
(5) x is called a **minimum of** A if $x \in A$ and x is a lower bound for A. Again we will often write $x = \min A$.

Previously we said a sequence, (a_n), was bounded if the associated set, $\{a_n : n \in \mathbb{N}\}$ was contained in a bounded interval. How do these two uses of the same word relate to each other?

Note that if a set has a maximum or a minimum, it only has one of each (why?)

3.58. For each of the following subsets of $\mathbb{R}$:

(a) $A = \varnothing$,
(b) $A = [0, 1]$,
(c) $A = (0, 1) \cap \mathbb{Q}$.
(d) $A = [0, \infty)$,

(1) Find all lower bounds for A and all upper bounds for A,

(2) Discuss whether A is bounded.
(3) Discuss whether A has a maximum and if so find it.
(4) Discuss whether A has a minimum and if so find it.

In particular, we see that not every set has a maximum and so the usefulness of the concept is a bit limited. Thus we introduce a more useful term which captures a similar idea.

Definition. Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

(1) x is called a **least upper bound of** A or **supremum of** A if
  (a) x is an upper bound for A,
  (b) for all y, if y is an upper bound for A then $x \leq y$.
(2) x is called a **greatest lower bound of** A or **infimum of** A if
  (a) x is a lower bound for A,
  (b) for all y, if y is a lower bound for A then $y \leq x$.

3.59. Let $A \subseteq \mathbb{R}$. Prove that if x is a supremum of A and y is a supremum of A then $x = y$.

Hence, if the set A has a supremum then that supremum is unique and we can speak of *the* supremum of A and write $\sup A$. Similarly, if the set A has an infimum then that infimum is unique and we can speak of *the* infimum of A and write $\inf A$. The supremum is a generalization of the idea of a maximum and the infimum is a generalization of the idea of a minimum.

3.60. Suppose that $A \subset \mathbb{R}$ is a set and that $\max A$ exists. Show that $\sup A$ exists and $\max A = \sup A$. Similarly if $\min A$ exists show that $\inf A$ exists and $\min A = \inf A$.

To better understand the infimum and the supremum, we should consider some examples.

3.61. Find $\inf A$ and $\sup A$, if they exist, for each of the following subsets of $\mathbb{R}$:

(1) $A = \varnothing$,
(2) $A = [0, 1]$,
(3) $A = (0, 1) \cap (\mathbb{R} \backslash \mathbb{Q})$
(4) $A = [0, \infty)$.

Thus we see that, like the maximum, the supremum also does not always exist. On the other hand, let us consider the reasons why the supremum failed to exist in some of the previous examples. Firstly, if the set in question is the empty set it cannot have a supremum. Hence assume that it is nonempty. We saw in the previous exercise a nonempty set without an supremum. What was it about that set which forced it not to have a supremum?

In fact this is the only other reason a supremum might not exist.

3.62. Suppose that A is nonempty and bounded above. Then A has a supremum.

Hint: Show that there is a least element $k_1 \in \mathbb{Z}$ such that k_1 is an upper bound for A. If k_1 is not a least upper bound for A, show there is a least $k_2 \in \mathbb{N}$ such that $k_1 - \frac{1}{2^{k_2}}$ is an upper bound for A. Proceed in this way to find the supremum.

Again we needed to use the completeness axiom in the proof of the previous result. Like the convergence of Cauchy sequences, the truth of the last result is equivalent to the completeness axiom. In fact, most authors take the last result as their formulation of the completeness axiom.

We have a similar existence result for infimums.

3.63. Suppose that A is nonempty and bounded below. Then $\inf A$ exists.

3.64. Suppose that $A \neq \varnothing$ is bounded. Prove that $\inf A \leq \sup A$.

3.65. Give examples, if possible, of the following.

(1) A set A with a supremum but no maximum.
(2) A decreasing sequence $(a_n)_{n=1}^{\infty}$ so that

$$\inf\{a_n | n \in \mathbb{N}\}$$

does not exist.
(3) An increasing sequence $(a_n)_{n=1}^{\infty}$ so that

$$\inf\{a_n | n \in \mathbb{N}\}$$

does not exist.

Now that we have proven the existence of supremums and infimums for nonempty bounded sets, we will study their properties. For example, though $\sup A$ need not be in the set A there are elements of A arbitrarily close to $\sup A$.

3.66. Let $A \subseteq \mathbb{R}$ be non-empty and bounded above. Prove that if $s = \sup A$ then for every $\epsilon > 0$ there exists an $a \in A$ with $s - \epsilon < a \leq s$.

3.67. What result regarding the infimum would correspond to the previous result? State and prove it.

3.68. Let $A \subseteq \mathbb{R}$ be bounded above and non-empty. Let $s = \sup A$. Prove that there exists a sequence $(a_n) \subseteq A$ with $\lim_{n\to\infty} a_n = s$. Prove that, in addition, the sequence (a_n) can be chosen to be increasing.

3.69. Again state and prove an analogous result for infimums.

3.70. Prove or disprove:

(1) If $A, B \subseteq \mathbb{R}$ are nonempty sets such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$ then $\sup A$ and $\inf B$ exist and $\sup A \leq \inf B$.
(2) If $A, B \subseteq \mathbb{R}$ are nonempty sets such that for every $a \in A$ and for every $b \in B$ we have $a < b$, then $\sup A$ and $\inf B$ exist and $\sup A < \inf B$.

3.71. Let $A \subset \mathbb{R}$ be nonempty and suppose that $A \subset [a, b]$. Show that $\sup A$ and $\inf A$ exist and that

$$a \leq \inf A \leq \sup A \leq b.$$

3.72. Let $A, B \subset \mathbb{R}$ be non-empty, bounded sets. We define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Prove that $\sup(A + B) = \sup(A) + \sup(B)$. State and prove a similar result regarding infimums.

3.73. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and $A \subseteq \mathbb{R}$, with $A \neq \varnothing$. Assume that $f(A)$ and $g(A)$ are bounded. Prove that

$$\sup(f + g)(A) \leq \sup f(A) + \sup g(A) .$$

Give an example where one has equality in the inequality and an example where one has strict inequality.

3.74. Let $A \subset \mathbb{R}$ be a non-empty, bounded set. Let $\alpha = \sup(A)$ and $\beta = \inf(A)$, and let $(a_n)_{n=1}^{\infty} \subset A$ be a convergent sequence, with $a = \lim_{n\to\infty} a_n$. Prove that $\beta \leq a \leq \alpha$.

3.75. In the notation of the previous result, give an example where the sequence (a_n) is strictly increasing and yet $a \neq \alpha$. Give an example where the sequence is strictly decreasing and yet $a \neq \beta$.

3.76. If $A \subset \mathbb{R}$ is a non-empty, bounded set and $B \subset A$ is nonempty, prove

$$\inf(A) \leq \inf(B) \leq \sup(B) \leq \sup(A) \ .$$

3.77. If $A, B \subset \mathbb{R}$ are both non-empty, bounded sets, prove

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\} \ .$$

8. Real and Rational Exponents

In calculus, we often consider functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = 2^x$. To understand what the expression 2^x means we need to define a notion for exponentiating by an arbitrary real number, rather than just by integers. We will do so in this section, beginning with exponentiating by rational numbers.

3.78. Suppose that $a < b$ are positive real numbers. Show that, if k is any natural number (not including zero), $a^k < b^k$.

Definition. Suppose that $a \geq 0$ and $k \in \mathbb{N}$. We say that x is a **kth root** of a if $x \geq 0$ and $x^k = a$. The previous result shows that that kth roots are unique. (why?) We denote the kth root of a by $\sqrt[k]{a}$ or $a^{1/k}$.

3.79. Suppose that $x, a \in \mathbb{R}$ satisfy $x^k < a$. Show that there is a number $y \in \mathbb{R}$ so that $x < y$ and yet $y^k < a$. If $x^k > a$ show that there is a number $y \in \mathbb{R}$ such that $x > y$ and yet $y^k > a$.

Hint: For the first part, consider the sequence $(x + \frac{1}{n})^k$. To what real number does it converge? What does that tell you?

The next result demonstrates a powerful application of supremums.

3.80. Let $a \geq 0$ and $k \in \mathbb{N}$. Show that $\sqrt[k]{a}$ exists in $\mathbb{R}$.

We point out that the roots are a purely algebraic concept and yet the completeness of $\mathbb{R}$ implies their existence. This is a very special property of $\mathbb{R}$. We have seen for example that it does not hold for $\mathbb{Q}$.

3.81. Suppose that $r \in \mathbb{Q}$ and $a \geq 0$. Give a definition for a^r. Show that it agrees with the special cases when $r \in \mathbb{N}$ or $r = 1/k$ for some $k \in \mathbb{N}$. Prove the usual rules of exponents:

(1) $a^{sr} = (a^r)^s$ for $r, s \in \mathbb{Q}$ and $a \geq 0$,
(2) $a^{s+r} = a^r a^s$ for $r, s \in \mathbb{Q}$ and $a \geq 0$, and
(3) $a^{-r} = \dfrac{1}{a^r}$ for $r \in \mathbb{Q}$ and $a \geq 0$.

Of course we can also consider roots for negative numbers.

3.82. Suppose that $a \in \mathbb{R}$ (a not necessarily positive), for which k is there a real number x with $x^k = a$. Give the values for all such x in terms of $\sqrt[k]{|a|}$, which we have already defined.

Hint: Of course we know that if k is even and $a < 0$, there is no $x \in \mathbb{R}$ with $x^k = a$. Proving it, however, is not completely trivial. Derive a contradiction by deciding whether such an x would have to be positive or negative.

We now lay the ground work for exponentiating by real numbers.

3.83. Suppose that $a > 0$ and show that $(\sqrt[n]{a})_{n=1}^{\infty}$ converges to one.

Hint: Begin with the case $a > 1$.

3.84. Suppose that (r_n) is a Cauchy sequence of rational numbers. Show that if $a \geq 0$, (a^{r_n}) is a also Cauchy.

3.85. Suppose that (r_n) and (s_n) are two sequences of rational numbers which converge to the same real number r. Show that $\lim\limits_{n\to\infty} a^{r_n} = \lim\limits_{n\to\infty} a^{s_n}$.

Definition. Suppose that $r \in \mathbb{R}$ and $a \geq 0$. Pick a sequence (r_n) of rational numbers that converges to r. Then we define a^r to be $\lim_{n\to\infty} a^{r_n}$.

3.86. Show that the normal rules of exponents hold for real exponents as well as rational ones. Show that if $a > 0$, the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a^x$ is injective.