CHAPTER 4

Limits of Functions and Continuity

1. Limits of Functions

In the previous chapter, we used the notion of convergence to refine our understanding of the real numbers. In this chapter, we use many of these same intuitive notions to study *functions*. First a word about conventions. In the section in which we introduced functions, we were very careful to specify that a function has three ingredients: a domain, a codomain, and a rule (which is technically a certain subset of the cartesian product of the domain and codomain).

For the remainder of the book, however, (unless we specify otherwise) the codomain of our functions will always be $\mathbb{R}$ itself. Likewise the domain will always be some subset of $\mathbb{R}$. Thus when we say f is a *function*, we mean that it is a function whose codomain is $\mathbb{R}$ and whose domain is some (potentially unspecified) subset of $\mathbb{R}$.

Furthermore, though it is also technically incorrect, we will often specify functions by giving only a formula or rule (as calculus textbooks often do). For example, we might say "consider the function $f(x) = 1/x$." This is technically not enough information. In this case, there are a large variety of subsets of $\mathbb{R}$ which could serve as the domain. Nevertheless when we say something like the above, we really mean that the domain should be every real number for which the formula makes sense. In our previous case then, the domain would be $\mathbb{R}\backslash\{0\}$, that is, all real numbers except for zero.

At times, this sort of vagueness can get one into trouble, but in all the cases we will consider, the intended domain will be clear. In fact, it is a (perhaps unfortunate) common practice in mathematics for an author, when defining an object, to specify only the noteworthy piece of data and leave it up to the reader to surmise the other pieces. In the case of functions, this is the practice taken by most calculus books (though perhaps without warning) and it will be our practice as well.

Now that we have made our conventions clear, we proceed to discuss limits of functions. Intuitively, we saw that a sequence (a_n) has limit L if it gets arbitrarily close to L as n gets larger. The intuitive idea of a function is similar, but whereas sequences themselves (possibly) have limits, functions (possibly) have limits *at points* in $\mathbb{R}$. In other words, if we want to consider the limit of a function, we need to specify the point at which we are focusing our attention.

Intuitively then, a function f has limit L at the point $p \in \mathbb{R}$ if the value, $f(x)$, of the function gets arbitrarily close to L as x gets very close to p. In particular, the limit does not depend on the value of f at p but only on the value of f at points x *near* p. Indeed, for a limit to exist at p it is not even necessary that f be defined at p. It is, however, necessary that f be defined at points x *near* p, in a sense made precise below.

Definition. Let f be a function and $L, p \in \mathbb{R}$. We say that L is a **limit of f as x approaches p** if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $0 < |x - p| < \delta$, x is in the domain of f and $|f(x) - L| < \epsilon$.

Again you should think careful about this definition. How is closeness to L measured? How is closeness to p measured? Where does $f(x)$ need to be defined? How does δ relate to ϵ? How does this definition compare to the notion of a limit that we have described above? How does it compare to the notion of a limit that you developed in your calculus courses? You should also try to draw some pictures of the situation. Where should ϵ go? What about δ?

Notice that when proving that L is a limit of f as x approaches p, we are given an arbitrary $\epsilon > 0$ and have to find a $\delta > 0$ exactly as we had to find a $N \in \mathbb{N}$ when proving that L was the limit of a sequence. In your proofs, you will need to choose δ judiciously, depending upon what f, p, and ϵ are.

Exactly as for sequences, we have to show that limits of functions are in fact unique.

4.1. Let f be a function and $p \in \mathbb{R}$. Suppose L and M are both limits of f as x approaches p. Show that $L = M$.

This shows that if a limit of f as x approaches p exists then it is unique. As with sequences, we can talk about *the* limit of f as x approaches p and write $\lim\limits_{x \to p} f(x) = L$.

Similar to sequences, we should not write $\lim\limits_{x \to p} f(x)$ unless we know that limit exists. In other words, as with sequences, when we write

$\lim_{x \to p} f(x) = L$ we are making two assertions: the limit of f as x approaches p exists, and its value is L.

4.2. Let f be a function and $p, L \in \mathbb{R}$. Give the negation of the definition of "$\lim_{x \to p} f(x) = L$".

Again the negation of "$\lim_{x \to p} f(x) = L$" is not "$\lim_{x \to p} f(x) \neq L$" since the latter implies implies the existence of the limit. The negation could read "$f(x)$ does not approach L as x approaches p." For this there are two possibilities: $\lim_{x \to p} f(x)$ exists but does not equal L, or f has no limit as x approaches p, but try to give the negation in terms of ϵ and δ as in the definition.

4.3. Let f be a function and $p, L \in \mathbb{R}$. Assume there exists $\epsilon_0 > 0$ so that x is in the domain of f if $0 < |x - p| < \epsilon_0$. Determine which, if any, of the following are equivalent to $\lim_{x \to p} f(x) = L$.

(1) For all $\epsilon > 0$, there is a $\delta > 0$ and an x with $0 < |x - p| < \delta$ and $|f(x) - L| < \epsilon$.
(2) For all $n \in \mathbb{N}$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $0 < |x - p| < \delta$, we have $|f(x) - L| < 1/n$.
(3) For all $n \in \mathbb{N}$, there exists a $m \in \mathbb{N}$ such that for all $x \in \mathbb{R}$ with $0 < |x - p| < 1/m$, we have $|f(x) - L| < 1/n$.
(4) For all $\delta > 0$ there is an x in the domain of f such that $0 < |x - p| < \delta$ and $f(x) = L$.
(5) There exists a $\delta > 0$ such that for all $\epsilon > 0$, if x is in the domain of f and $0 < |x - p| < \delta$ then $|f(x) - L| < \epsilon$.

As usual, an important first step in understanding a new concept (or a least a concept that we are meeting rigorously for the first time), is considering examples. We begin with the simplest functions: constant functions.

4.4. Let $a \in \mathbb{R}$ and let f be the function given by $f(x) = a$ for all $x \in \mathbb{R}$. Let $p \in \mathbb{R}$ be arbitrary. Prove that $\lim_{x \to p} f(x) = a$.

Notice that, in the case of a constant function, you can choose a δ that does not depend on ϵ. This behavior is extremely rare for a function.

We now proceed with a few more examples.

4.5. Let f be the function given by $f(x) = x$ and let $p \in \mathbb{R}$. Prove $\lim_{x\to p} f(x) = p$.

Now we see that δ depends on ϵ and that δ goes to 0 as ϵ goes to 0. However the choice of δ still does not depend on the choice of p.

4.6. Let f be the function given by $f(x) = 3x-5$ and let $p \in \mathbb{R}$. Prove $\lim_{x\to p} f(x) = 3p - 5$.

4.7. Let f be the function given by $f(x) = x/x$. Show that $\lim_{x\to 0} f(x) = 1$, despite the fact that f is not defined at zero.

The previous example shows that indeed a function need not be defined at a point to have a limit there. In fact, this is even more obvious than we are making it seem: we can always take a function that has a limit at some point and then create a new function by removing this point from the domain. This new function will have the behavior to which we are referring. The previous example is, however, interesting since it is an example of a function of this kind defined by a simple formula.

4.8. Let f be the function given by $f(x) = x^2$ and let $p \in \mathbb{R}$. Prove $\lim_{x\to p} f(x) = p^2$.

Notice that regardless of how we produce our choice of δ for the previous proof, it always depends on the point p. For a fixed ϵ we see that the δ required gets smaller and smaller as $|p|$ gets bigger and bigger. How could you tell this from looking at the graph of f?

4.9. Prove that $\lim_{x\to 2}(2x^2 - x + 1) = 7$.

4.10. Let f be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational.} \end{cases}$$

Prove that $\lim_{x \to p} f(x)$ does not exist for any $p \in \mathbb{R}$.

4.11. Let f be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ x & x \text{ is rational.} \end{cases}$$

Prove that $\lim_{x \to p} f(x)$ exists for only one value of p.

We now add a brief discussion regarding the domain of a function for which we want to consider a limit at $p \in \mathbb{R}$. Indeed, one of the requirements of the definition of a limit says that we must find a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $0 < |x - p| < \delta$, x is in the domain of f. In other words, the set $(p - \delta, p + \delta) \backslash \{p\}$ must be contained in the domain of f. Hence we see that f is defined on $I \backslash \{p\}$, where I is an open interval containing p. It is thus convenient to make the following definition.

Definition. Let $p \in \mathbb{R}$. We say a function f is **defined near** p if the domain of f contains a set of the form $I \backslash \{p\}$ where I is an open interval containing p.

In other words f is defined near p if there is some $\delta > 0$ such that for $x \in \mathbb{R}$ with $0 < |x - p| < \delta$, x is in the domain in f. Essentially that means that we can find a small sliver of the real line containing p on which f is defined (except for possibly at p). Do you think that this is a good meaning for the phrase 'near p'? Why or why not?

4.12. Suppose $p, L \in \mathbb{R}$ and that f is a function. Show that $\lim_{x \to p} f(x) = L$ if and only if f is defined near p and for every $\epsilon > 0$ there is a $\delta > 0$ such that for $x \in \mathbb{R}$ with $0 < |x - p| < \delta$, $|f(x) - L| < \epsilon$ whenever $f(x)$ is defined.

The preceding discussion and result had very little content to it and was essentially semantics or technicalities. Nevertheless, though we would all prefer to avoid them, technicalities play an essential role in the study of mathematics. In particular, the discussion above is important because of certain complications that might be introduced if were less careful.

For example, suppose that we had defined "$f(x)$ converges to L as x approaches p" to mean "for every $\epsilon > 0$ there is a $\delta > 0$ such that for

$x \in \mathbb{R}$ with $0 < |x - p| < \delta$, $|f(x) - L| < \epsilon$ whenever $f(x)$ is defined." We just saw that this definition is equivalent to ours, provided that f is defined near p. If, however, the domain of f contained no points in an open interval containing p, then *any* real number would satisfy this definition of limit at p! Clearly this means that the definition above is not a good one. We conclude that our definition needs to force f to be defined near p (or at least at some points near p: some authors use a slightly more general definition, but we have used ours to avoid even more technicalities).

More generally, we say a property of functions is true **near** p if it is true on a set of the form $I - \{p\}$ with I an open interval containing p. For example we might say $f(x) \leq g(x)$ is true near p to say that $f(x) \leq g(x)$ is true for all x on some set of the form $I \backslash \{p\}$ (in particular we are asserting that both f and g are defined near p).

Now that we have considered a few examples and thought a little bit about the above definition, the next logical step is to study its basic properties. Before we do so however, we introduce another important concept which will help us to better understand limits. It is the notion of a 'one-sided limit'. Intuitively a one-sided limit describes the situation when we approach the number p from only one side rather then from either side.

Definition. Let f be a function and $p, L \in \mathbb{R}$. We say that L is a **right-hand limit of f as x approaches p** if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$ with $p < x < p + \delta$, x is in the domain of f and $|f(x) - L| < \epsilon$. Similarly, we say that L is a **left-hand limit of f as x approaches p** if for every $\epsilon > 0$ there exists a $\delta > 0$, such that for $x \in \mathbb{R}$ with $p - \delta < x < p$, x is in the domain of f and $|f(x) - L| < \epsilon$.

4.13. Show that left-hand and right-hand limits are unique.

The left-hand limit is denoted by $\lim_{x \to p^-} f(x) = L$ and the right-hand limit is denoted by $\lim_{x \to p^+} f(x) = L$.

Left-hand and right-hand limits also require something about the domain of the function f. We might say that a right-hand limit requires f to be defined "near p on the right" and likewise for left-hand limits.

4.14. Give an example of a function f and a point $p \in \mathbb{R}$ such that $\lim_{x \to p^+} f(x)$ exists but $\lim_{x \to p^-} f(x)$ does not. Give an example of a function and a point where the left-hand limit and the right-hand limit both exist but they are not equal.

4.15. Let f be a function and $p \in \mathbb{R}$. Prove that $\lim_{x \to p} f(x) = L$ if and only if $\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = L$.

Thus in order for the limit at p to exist it is necessary and sufficient for the right-hand and left-hand limits to exist and be equal. Now that we have considered one-sided limits, we proceed to study the basic properties of limits of functions. Our first task is give a relationship between limits of functions and limits of sequences. Indeed, the next result is known as the **sequential characterization of limits** (though it might be more precise to say that it is the "sequential characterization of limits of functions.")

4.16. Let $p, L \in \mathbb{R}$ and suppose that f is a function defined near p. Let D be the domain of f. Prove that $\lim_{x \to p} f(x) = L$ if and only if for every sequence $(x_n) \subseteq D \setminus \{p\}$, with $x_n \to p$, we have $\lim_{n \to \infty} f(x_n) = L$.

Hint: To prove "if for every sequence $(x_n) \subset D \setminus \{a\}$ with $\lim_{n \to \infty} x_n = p$ we have $\lim_{n \to \infty} f(x_n) = L$ implies that $\lim_{x \to p} f(x) = L$" you should switch to the contrapositive.

Why did we have to assume that f was defined near p in the previous result?

4.17. a) State the contrapositive of the sequential characterization of limits (i.e., get a new if and only if statement by negating both sides).

b) Let $f : D \to \mathbb{R}$ be defined near a point $p \in \mathbb{R}$. Prove that the limit of f as x approaches p does not exist if and only if there exists a sequence $(x_n) \subseteq D \setminus \{p\}$ with $x_n \to p$ so that $(f(x_n))_{n=1}^{\infty}$ diverges.

This statement is quite useful for proving that a function has no limit as x approaches p.

4.18. Which of these limits (if any) exist? Prove your answer.

(1) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$.
(2) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

Hint: By definition $\sin(\theta)$ is the y-coordinate of the point on the unit circle at angle θ from the positive x-axis. You will be able to complete this exercise using only this definition (and the results of the course thus far).

We now prove the Limit Laws for functions (which are of course analogous to the limit laws for sequences). As in the sequential cause, we need to begin with a preliminary result which will help us.

4.19. Let f be a function let $p \in \mathbb{R}$. Assume $\lim_{x\to p} = L$ and $L > 0$. Prove $f(x) > L/2$ near p.

Now we prove the Limit Laws. If f and g are functions, we remark the the formulas $f(x) + g(x)$ and $f(x)g(x)$ make sense as long as x is in both the domain of f and the domain of g and so the domain of the functions given by these formulas is defined to be the intersection of the domain of f with that of g. Likewise, the domain of the function given by $f(x)/g(x)$ is the intersection of the domain of f and the set on which g is nonzero.

4.20. Let $p \in \mathbb{R}$ and let f and g be functions satisfying

$$\lim_{x\to p} f(x) = L \qquad \lim_{x\to p} g(x) = M$$

Let $c \in \mathbb{R}$. Prove that

(1) $\lim_{x\to p} c \cdot f(x) = c \cdot L$.
(2) $\lim_{x\to p} \big(f(x) + g(x)\big) = L + M$.
(3) $\lim_{x\to p} \big(f(x) \cdot g(x)\big) = L \cdot M$.
(4) If g is nonzero near p and $M \neq 0$ then $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Hint: There are two ways to prove these statements; one is to use the definition of limit directly as with sequences, and the other is to use the sequential characterization of limits and the Limit Laws for sequences.

As with sequences, we have a result discussing the interplay between the order on $\mathbb{R}$ and limits of functions.

4.21. Let f be a function and $p \in \mathbb{R}$. Assume that $a \leq f(x) \leq b$ near p. Prove that if $L = \lim_{x\to p} f(x)$, then $L \in [a, b]$.

We also have the Squeeze Theorem for functions.

4.22. Let f, g, and h be functions and let $p \in \mathbb{R}$. Suppose that $g(x) \leq f(x) \leq h(x)$ near p. Prove that if $\lim_{x\to p} g(x) = \lim_{x\to p} h(x) = L \in \mathbb{R}$, then $\lim_{x\to p} f(x) = L$.

2. Continuous Functions

Hopefully your calculus courses convinced you that continuity is an important property. At some point, you probably heard your instructor say something to the effect of "a function is continuous if you can draw its graph without picking up your pencil." In some sense this is the intuitive idea behind continuity, but in this case we will find the precise definition leads us a little further from the intuitive notion than in the previous cases we have considered.

With a few technicalities to be considered, to say that a function f is continuous at $p \in \mathbb{R}$ is to say that p is in the domain of f (which is not a requirement to consider the limit of the function at p) and the value of $f(x)$ gets arbitrarily close to $f(p)$ as x gets close to p.

Definition. Let f be a function and let p be in the domain of f. We say that **f is continuous at p** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for $x \in \mathbb{R}$ with $|x-p| < \delta$, we have $|f(x)-f(p)| < \epsilon$ whenever x is in the domain of f.

4.23. Let f be a function and let p be in the domain of f. Negate the definition of "f is continuous at p".

As always, we begin by considering examples

4.24. Show that the function $f(x) = x$ is continuous for every $p \in \mathbb{R}$.

4.25. Let f be given by

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find all points $p \in \mathbb{R}$ at which f is continuous. Justify your answer.

4.26. Let f be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational.} \end{cases}$$

Find all points $p \in \mathbb{R}$ at which f is continuous. Justify your answer.

4.27. Let f be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ x & x \text{ is rational.} \end{cases}$$

Find all points $p \in \mathbb{R}$ at which f is continuous. Justify your answer.

The next result can help add to our intuition of continuity.

4.28. Suppose that f is function and p is in its domain. Show that f is continuous at p if and only if for each open interval I containing $f(p)$, there is an open interval, J, containing p such that $f(x) \in I$ whenever x is in the domain of f and $x \in J$.

Thus we might say that f is continuous at p if, for x near p, $f(x)$ is always near (or equal to) $f(p)$ (when defined).

We will see momentarily that continuity and limits of functions are closely related (as we might expect by comparing their definitions). However there are some notable differences.

4.29. Define a function $f : \{0\} \to \mathbb{R}$ by putting $f(0) = 1$. Show that f is continuous at 0. Is the same true if $f : \{0\} \cup [1, 2] \to \mathbb{R}$ is given by $f(0) = 1$ and $f(x) = x$ for $x \in [1, 2]$.

The function above is certainly not defined near zero and yet it is continuous at zero. A difference then between our definition of continuity at a point and our definition of the limit at a point then is that the former does not require the function to be defined near the point. There are certain important reasons that we want to do it this way (mostly to line up with notions from higher mathematics courses such as 'topology'), but other authors may use a different convention.

We now make the connection between limits and continuity explicit.

4.30. Let f be a function and let p be in the domain of f. Assume that f is defined near p. Prove that f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

There is also a connection between limits of sequences and continuity. Not surprisingly, it is called the **sequential characterization of continuity** and it is the next result. In proving it, you should be careful to consider what happens if f is not defined near p.

4.31. Let f be a function, let D be the domain of f and let $p \in D$. Prove that f is continuous at p if and only if for every sequence $(x_n) \subseteq D$ with $\lim_{n \to \infty} x_n = p$, we have $\lim_{n \to \infty} f(x_n) = f(p)$.

4.32. Give the contrapositive to the sequential characterization of continuity.

Not surprisingly, we have results for continuity which are analogous to the Limit Laws.

4.33. Let f and g be functions and let p be a real number in the domain of each. Assume that f and g are continuous at p. Let $c \in \mathbb{R}$. Prove:

(1) $f + g$ is continuous at p.
(2) $c \cdot f$ is continuous at p.
(3) $f \cdot g$ is continuous at p.
(4) If $g(p) \neq 0$ then $\frac{f(x)}{g(x)}$ is continuous at p.

In the above problem you should consider the domains of the various functions. For example the domain of $f + g$ is $\{x : x \in \text{dom}(f) \cap \text{dom}(g)\}$. For the next problem, what is the domain of $g \circ f$?

4.34. Let f and g be functions and let p be a point in the domain of f such that $f(p)$ is in the domain of g. Prove that if f is continuous at p and g is continuous at $f(p)$ then $g \circ f$ is continuous at p.

This can be proved either directly from the definition or by repeated application of Problem 4.31.

So far, we have been discussing continuity only at points, but it is probably far more important to consider continuity on sets.

Definition. Let f be a function and S a subset of $\mathbb{R}$. We say that f is **continuous on *S*** if S is contained in the domain of f and if for each $p \in S$ and $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - p| < \delta$, we have $|f(x) - f(p)| < \epsilon$ whenever $x \in S$. We say a function is **continuous** if it is continuous on its domain.

We remark that the previous definition is used most frequently in the case that S is an interval (although not always). It is also a bit subtle. For example there is a difference between being continuous on S and being continuous at every point of S.

4.35. Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \in [0, 1] \\ 1 & x \in (1, \infty). \end{cases}$$

Show that f is continuous on $[0, 1]$ and yet it is not continuous at every point of $[0, 1]$. What part of the definition of 'continuous on S' allows this to be the case? What is

$$\{x \in [0, \infty) : f \text{ is continuous at } x\}?$$

Assuming we rid ourselves of some technicalities, however, the two notions do indeed coincide.

4.36. Suppose that S is a subset of $\mathbb{R}$ and f is a function. Assume in addition that S is the domain of f. Show that f is continuous on S if and only if it is continuous at every point of S.

In other words, if f is a function whose domain contains a set S, f is continuous on S if and only if $f|_S$ is continuous.

4.37. Let f and g be functions and let S be a subset of $\mathbb{R}$. Suppose that f and g are continuous on S. Let $c \in \mathbb{R}$. Prove:

(1) $f + g$ is continuous on S.
(2) $c \cdot f$ is continuous on S.
(3) $f \cdot g$ is continuous on S.
(4) $\frac{f(x)}{g(x)}$ is continuous on S, provided that g is never zero on S.

The terminology here is perhaps also sightly different than the terminology one would see in a calculus course.

4.38. Show that function $f(x) = 1/x$ is continuous.

Of course a calculus student would never say $f(x) = 1/x$ is continuous. What we are really saying is that it is continuous on its domain, that is, it is continuous on the set $(-\infty, 0) \cup (0, \infty)$. Calculus students would agree that this is the appropriate result. It is indeed false that the function is continuous on $\mathbb{R}$ because it is not even defined on all of $\mathbb{R}$.

Definition. We recall that a **polynomial** is a function on $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for an integer $n \geq 0$ and $a_i \in \mathbb{R}$. If f is not the zero function (that is, the function whose every value is zero), we may assume $a_n \neq 0$ and we call n the **degree** of f. A **linear function** is a function which is either a polynomial of degree zero or one or a function which is identically zero. Polynomials of degrees 2, 3, 4, and 5 are called **quadratic functions**, **cubic functions**, **quartic functions**, and **quintic functions**, respectively.

Definition. A **rational function** is a function of the form $f(x) = g(x)/h(x)$ where g and h are polynomials (so that the domain of f is the set where h is nonzero).

4.39. Prove that every polynomial is continuous on $\mathbb{R}$ and that every rational function is continuous.

4.40. Let $a \geq 0$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a^x$ is continuous.

3. Theorems About Continuous Functions

One of the fundamental theorems regarding continuous functions, more specifically functions which are continuous on an closed intervals, is the **Intermediate Value Theorem**.

We give the explicit formulation next, but first we discuss the intuitive statement. Suppose that f is a function and we know that f is continuous on $[a, b]$. Intuitively, we can draw the graph of f (at least between a and b) without picking up our pencil. Hence if y is some value between the beginning value, $f(a)$, of f and the ending value, $f(b)$, of f, we should expect that at some point the graph of f should have to 'cross' the value y. In other words, there should be a $c \in [a, b]$ with $f(c) = y$.

4.41. Suppose $a < b$ and f is a function which is continuous on $[a, b]$. If $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$ then there exists $c \in (a, b)$ with $f(c) = y$.

Hint: Suppose $f(a) < y < f(b)$ and let $E = \{x \in [a, b] : f(x) < y\}$. Let $p = \sup E$. Show the point p can be written as the limit of a sequence $x_n \in E$ and as the limit of a sequence $x'_n \in [a, b] \setminus E$. Then prove that $f(p) = y$.

Notice that in proving the previous result, we needed to use the completeness axiom. This is not a coincidence as should perhaps be intuitively clear. Indeed, if there were 'gaps' in the real line than those gaps would allow a function to 'jump over' the value y without actually hitting it.

4.42. Suppose that f is a polynomial of odd degree. Show that f has a zero. That is, show that there is some $a \in \mathbb{R}$ with $f(a) = 0$.

4.43. Suppose that $a \geq 0$ and suppose $k \in \mathbb{N}$. Use the intermediate value theorem to give another proof that $\sqrt[k]{a}$ exists.

4.44. Suppose that $a, b > 0$. Show that there is a unique number $c \in \mathbb{R}$ so that $a^c = b$.

Definition. If $a, b \geq 0$, the unique number c with $a^c = b$ is called the **logarithm base *a* of** b and denoted $\log_a(b)$.

4.45. Show the usual rules of logarithms hold.

We next study the images of intervals under continuous functions.

4.46. Let $I \subseteq \mathbb{R}$ be any interval and suppose that f is a function which is continuous on I. Furthermore, assume that f is nonconstant on I (meaning that f takes on more than one value on I). Prove that $f(I)$ is an interval.

Hint: First prove that it suffices to show that given any two points $c, d \in f(I)$ the entire interval between them is contained in $f(I)$.

In general we cannot say any more about the interval $f(I)$. In other words, it might be bounded or unbounded and it might be open or closed or neither.

Definition. Let S be a subset of R and let f be a function. Then we say that f is **bounded** on S if S is contained in the domain of f and the set $f(S)$ is bounded. Thus if f is defined on S, it is bounded on S if and only if there exists $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in S$ (why?).

4.47. Give an example of a function which is continuous on $(0, 1)$ but not bounded on $(0, 1)$. Given an example of a function which is continuous on $(0, 1]$ but bounded neither above nor below on $(0, 1]$.

Thus the continuous image of a bounded interval may be unbounded.

4.48. Give an example of a continuous function $f : (0, 1) \to \mathbb{R}$ such that $f\big((0, 1)\big)$ is a closed and bounded interval.

Thus the continuous image of an open interval may be a closed interval.

However, in the special case of a continuous function on a closed and bounded interval we can say a lot more.

4.49. Let I be a closed bounded interval and suppose that f is a function which is continuous on I. Show that f is bounded on I.

Hint: Put $I = [a, b]$ and proceed by contradiction. Suppose that f is not bounded above on I and construct a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in [a, b]$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} f(x_n) = +\infty$$

Now apply the sequential characterization of continuity, Problem 4.31, to obtain a contradiction.

The next result is called the **Extreme Value Theorem**.

4.50. Let I be a closed and bounded interval and suppose that f is a function which is continuous on I. Show that there exist $x_m, x_M \in I$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$. In other words, f attains a **maximum value** and a **minimum value** on I.

Hint: Construct a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in [a,b]$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} f(x_n) = \sup\{f(x) : x \in [a,b]\}.$$

4.51. Give an example of a function which is bounded and continuous on $(0,1)$ but has neither a maximum nor a minimum on $(0,1)$. Can you do the same for $(0,1]$?

4. Uniform Continuity

In this section we discuss an important property for functions which is actually stronger than continuity. Recall that the function f is continuous on the set S if S is in the domain of f and if for all $p \in S$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ with $|x-p| < \delta$ we have $|f(x) - f(p)| < \epsilon$.

For a continuous function, the δ generally depends upon both ϵ and the point p as previous exercises have illustrated. If we remove the dependence on p, we get our new concept.

Definition. Let f be a function and S a subset of $\mathbb{R}$ contained in the domain of f. We say that f is **uniformly continuous** on S if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in S$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

4.52. Suppose $S \subseteq \mathbb{R}$ and let f be a function. Prove that if f is uniformly continuous on S than it is continuous on S.

4.53. Let f be a **linear function**. That is, let $f(x) = m\,x + b$ for some $m, b \in \mathbb{R}$. Prove that $f(x)$ is uniformly continuous on $\mathbb{R}$.

4.54. Negate the definition of uniform continuity.

4.55. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Prove that f is not uniformly continuous on $\mathbb{R}$.

Hint: Fix an $\epsilon > 0$ and show that no matter what $\delta > 0$ is chosen you can always choose $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|x^2 - y^2| \geq \epsilon$.

4.56. Prove that if I is a bounded interval and f is a function which is uniformly continuous on I then f is bounded on I.

This together with Problem 4.47 shows that we can have continuous functions on $(0, 1)$ that are not uniformly continuous (why?). As in the previous section the case of functions on closed and bounded intervals is very different.

4.57. Let I be a closed and bounded interval. Show that a function is continuous on I if and only it is uniformly continuous on I.

Hint: Suppose that f is not uniformly continuous. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists $x, y \in [a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. Show that there exists two sequences $(x_n), (y_n) \subset [a, b]$ which both converge to the same point $p \in [a, b]$ but such that $|f(x_n) - f(y_n)| \geq \epsilon$. Show that this implies that f is not continuous on I.