CHAPTER 5

Differentiation

1. Derivatives

Towards the beginning of Chapter 3, we stated that it is convergence and limits that separate calculus from algebra or other subjects. In fact, this is perhaps a bit misleading as convergence and limits really belong exclusively to a branch of mathematics known as topology (parts of which of course appear in calculus). Calculus is characterized by the derivative and the integral. In this chapter, we study the former.

Definition. Let f be a function and p a point in the domain of f. Then f is said to be **differentiable at p** if

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

exists. If f is differentiable at p then the above limit is called the **derivative of** f **at** p, and is usually denoted by $f'(p)$. If S is a subset of $\mathbb{R}$, f is said to be **differentiable on** S if f is differentiable at p for all $p \in S$ (which in particular implies that f is defined on S). f is called **differentiable** if it is differentiable on its domain.

Before we go on to give some examples, we will give a brief discussion the relationship between derivatives and the domain. We know that in order for the limit above to exist, the function

$$\frac{f(x) - f(p)}{x - p}$$

must be defined near p. We see that this will happen if and only of f is defined near p (why?). Thus an implicit requirement for f to be differentiable at p is that it has to be defined near p (and it is explicitly assumed to be defined *at* p)

Also, rather than always speaking of the derivative at individual points, we may, given a function f, define its **derivative** as a function. The domain of derivative is the set of points at which f is differentiable and for such a point p, the value of the derivative is $f'(p)$. For obvious reasons, the derivative is typically denoted by f'.

5.1. We give some examples of derivatives.

(1) Let $c \in \mathbb{R}$ be arbitrary and let f be defined by defined by $f(x) = c$. Prove that f is differentiable on $\mathbb{R}$ and that $f'(x) = 0$ for all $x \in \mathbb{R}$.
(2) Let f be defined by $f(x) = x$. Prove that f is differentiable on $\mathbb{R}$ and that $f'(x) = 1$ for all $x \in \mathbb{R}$.
(3) Let f be defined by $f(x) = 3x-7$. Prove that f is differentiable on $\mathbb{R}$ and that $f'(x) = 3$ for all $x \in \mathbb{R}$.
(4) Let f be defined by $f(x) = x^2 - x + 1$. Prove that f is differentiable on $\mathbb{R}$ and find $f'(x)$ for $x \in \mathbb{R}$.
(5) Let f be defined by $f(x) = |x|$. Prove that f is differentiable on $\mathbb{R} - \{0\}$, but not at zero. Find f'.

We can also relate the notion of f being differentiable at p with our earlier notion of continuity.

5.2. Let f be a function and let p be in the domain of f. Prove that f is differentiable at p if and only if there exists a function ϕ, defined near and at p, such that ϕ is continuous at p and

$$f(x) = f(p) + (x - p)\phi(x)$$

near p. Moreover, if such a ϕ exists then $f'(p) = \phi(p)$.

Now many of our results about differentiability of f will follow from our rules for continuous functions applied to ϕ.

5.3. Let f be a function and p a point in the domain of f. Prove that if f is differentiable at p then f is continuous at p.

In particular, many of the usual rules of differentiation, such as the Product Rule and the Quotient Rule come from Problem 4.37 after some simple algebra.

5.4. Let f and g be functions, $c \in \mathbb{R}$, and let p be a point in the domain of f and the domain of g. If f is differentiable at p and g is differentiable at p then

(1) $c \cdot f$ is differentiable at p and

$$(c \cdot f)'(p) = c \cdot f'(p).$$

(2) $f + g$ is differentiable at p and

$$(f + g)'(p) = f'(p) + g'(p).$$

(3) $f \cdot g$ is differentiable at p and
$$(f \cdot g)'(p) = f(p) \cdot g'(p) + f'(p) \cdot g(p).$$
(4) if $g(p) \neq 0$ then $\frac{f}{g}$ is differentiable at p and
$$\left(\frac{f}{g}\right)'(p) = \frac{g(p) \cdot f'(p) - f(p) \cdot g'(p)}{\big(g(p)\big)^2}.$$

Hint: You can prove these either directly from the definition or by writing $f(x) = f(p) + (x - p) \cdot \phi(x)$ and $g(x) = g(p) + (x - p) \cdot \psi(x)$ and using Problem 5.2. If you really want to help yourself understand, you should try both ways and decide which is easier or clearer.

We can also prove the power rule.

5.5. Suppose that n is a natural number. Define a function by $f(x) = x^n$. Show that f is differentiable for all points $p \in \mathbb{R}$ and $f'(p) = np^{n-1}$.

5.6. Suppose that n is a natural number. Define a function by $f(x) = x^{1/n}$. State the domain of f. Find f' (with proof) and state its domain.

Hint: For two real numbers a and b, check that
$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$
How does this help?

5.7. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function. Show that P is differentiable and find (with proof) its derivative.

5.8. Let
$$R(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$
be a rational function. Show that P is differentiable and find (with proof) its derivative.

There is one more standard differentiation rule: the Chain Rule.

5.9. Let f and g be functions. Let $p \in I$. Prove that if f is differentiable at p and g is differentiable at $f(p)$ then $g \circ f$ is differentiable at p and
$$(g \circ f)'(p) = g'\big(f(p)\big) \cdot f'(p).$$

Hint: Since g is differentiable at $f(p)$ there exists an appropriate function ψ such that

$$g(x) = g\big(f(p)\big) + \big(x - f(p)\big) \cdot \psi(x).$$

Replace x by $f(x)$ and then use the fact that $f(x) - f(p) = (x-p)\cdot\phi(x)$ for an appropriate ϕ.

5.10. Suppose that $r \in \mathbb{Q}$. Find the domain of $f(x) = x^r$. Find f' and its domain (with proof).

2. Theorems About Differentiable Functions

Perhaps the most important application of differentiation is optimization. Optimization is a word given to the process of finding the maximum or minimum value of a given function (often subject to one, or more, constraints).

Definition. Let f be a function. We say a point p in the domain of f is a **local maximum** for f if $f(x) \leq f(p)$ for x near p. Explicitly this means that we can find a $\delta > 0$ such that $f(x) \leq f(p)$ for all $x \in \mathbb{R}$ with $|x - p| < \delta$. Likewise, we say p is a **local minimum** if $f(x) \geq f(p)$ for x near p.

We begin with a result that relates local maxima and minima with differentiation.

5.11. Suppose that f is a function and p is either a local maximum or a local minimum of f. If f is differentiable at p, show that $f'(p) = 0$.

Hint: Assume p is a local maximum and consider the sign of

$$\lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} \text{ and } \lim_{x \to p^-} \frac{f(x) - f(p)}{x - p}.$$

In particular, we can derive the theory behind a technique for solving many routine calculus problems.

Definition. Suppose that f is a function. $x \in \mathbb{R}$ is called a *critical point* of f if either f is not differentiable at x or if $f'(x) = 0$.

5.12. Suppose that f is a function which is continuous on the interval $[a, b]$. Then f attains a maximum and a minimum on $[a, b]$ and each occurs either at a critical point of f or at an endpoint of the interval $[a, b]$ (that is at a or b).

The previous observations also allow us to an important result known as Rolle's Theorem. A version of the theorem was first stated by Indian astronomer Bhaskara in the 12th century however. The first proof, however, seems to be due to Michel Rolle in 1691.

5.13. Let $a < b$. Suppose that f is a function which is continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ with $f'(c) = 0$.

Hint: Show that f has either a maximum value or a minimum value at some $c \in (a, b)$.

An immediate consequence of Rolle's Theorem is the extremely important Mean Value Theorem. The Mean Value Theorem is used extensively in estimating the values of functions and for many other purposes (some of which we demonstrate below).

5.14. Let $a < b$. Suppose that f is a function which is continuous on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hint: Construct a linear function $l(x)$ with $l(a) = f(a)$, $l(b) = f(b)$ and consider $g(x) = f(x) - l(x)$.

We now give some consequences of the Mean Value Theorem.

5.15. Let $a < b$. Suppose f is a function which is continuous on $[a, b]$ and differentiable on (a, b). Assume the derivative of f is zero on (a, b). Then f is a constant on $[a, b]$. Conclude that the only functions of the form $f : \mathbb{R} \to \mathbb{R}$ whose derivatives are identically zero are constant.

5.16. Suppose that f and g are two functions which are differentiable on an open interval I. Suppose that that $f' = g'$ on I. Show that f and g differ by a constant. In other words, show that there is a $c \in \mathbb{R}$ with $g = f + c$.

5.17. Let $a < b$. Suppose that f is a function which is continuous on $[a, b]$ and differentiable on (a, b). Prove that

(1) if $f'(x) \geq 0$ for all $x \in (a, b)$ then f is increasing on $[a, b]$, i.e. if $a \leq x < y \leq b$, then $f(x) \leq f(y)$.
(2) if $f'(x) \leq 0$ for all $x \in (a, b)$ then f is decreasing on $[a, b]$, i.e. if $a \leq x < y \leq b$, then $f(x) \geq f(y)$.

5.18. Let $f(x) = ax^2 + bx + c$ be a quadratic function (so that in particular $a \neq 0$). Prove that if $a > 0$ then $f(x)$ has an unique absolute minimum. In other words, there is a $p \in \mathbb{R}$ such that $f(x) > f(p)$ for all $x \neq p$. Find p and $f(p)$. Likewise if $a < 0$, show that f has a unique absolute maximum and find it coordinates. Show that f has a root if and only if $b^2 - 4ac \geq 0$.

5.19. Suppose that $f(x) = ax^3 + bx^2 + cx + d$ is a cubic function. Suppose that $b^2 - 3ac < 0$. Show that f has exactly one zero.

Hint: We have already shown that f has a root. To show that it has at most one, argue by contradiction using the Mean Value Theorem (or more precisely Rolle's Theorem).