M 361K Spring 2011 (55380), Final

You may use your notes, your old homework, and our text, but no other resources. In particular, you may not work together on this exam. In writing up your proofs, you may freely use (without re-proving) any result which is given as a problem in the text (except for Problem 3 as noted below.)

Choose **four of Problems 1-5** to solve. Also solve **Problems 6-7**. Your solutions are due at or before **5pm Friday December 9**, at my office, RLM 9.134 (you can put them under the door if I am not there.)

Problem 1 (40 points).

(a) (40 points) Consider the sequence $a_n = \frac{1}{n^4-3} + 2$. Prove that $(a_n)_{n=1}^{\infty}$ converges.

Problem 2 (40 points).

Suppose $L, M \in \mathbb{R}$. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are two sequences such that $a_n \to L$ and $b_n \to M$. Define $c_n = \begin{cases} a_n & \text{if } n \text{ even,} \\ b_n & \text{if } n \text{ odd.} \end{cases}$

(a) (20 points) Suppose that $L = M$. Prove that $(c_n)_{n=1}^{\infty}$ converges.

(b) (20 points) Suppose that $L \neq M$. Prove that $(c_n)_{n=1}^{\infty}$ does not converge.

Problem 3 (40 points).

Suppose $p \in \mathbb{R}$. Suppose f and g are functions which are continuous at p.

(a) (40 points) Prove that $f + g$ is continuous at p. (For this problem you may *not* just quote the result of Problem 4.33 from the text.)

Problem 4 (40 points).

Give examples of the following. You need not prove that your examples satisfy the conditions.

(a) (10 points) A divergent sequence a_n such that the sequence a_n^2 is convergent.

(b) (10 points) A function f with domain $\mathbb{R}$, such that if $p \in \mathbb{R}$, f is continuous at p if and only if $p \geq 0$.

(c) (10 points) A continuous function f with domain $\mathbb{R}$, such that the image of f is the interval $[0, 1]$.

(d) (10 points) Two functions f and g with domain $\mathbb{R}$, such that f and g are not continuous on $\mathbb{R}$ but $f + g$ is continuous on $\mathbb{R}$.

Problem 5 (40 points).

In class we have proven that for any $x \in \mathbb{R}$ with $x \geq 0$ there is a unique number $\sqrt{x} \in \mathbb{R}$, such that $\sqrt{x} \geq 0$ and $(\sqrt{x})^2 = x$. Let $a_n = \sqrt{1 - 1/n}$.

(a) (20 points) Prove that the sequence $(a_n)_{n=1}^{\infty}$ converges. (Hint: one approach is to use the completeness axiom of $\mathbb{R}$.)

(b) (20 points) Prove that $\lim_{n\to\infty} a_n = 1$.

Problem 6 (40 points).

Suppose that f and g are functions with domain $\mathbb{R}$. Define a new function h by

$$h(x) = \max(f(x), g(x)).$$

Let $p \in \mathbb{R}$.

(a) (25 points) If both f and g are continuous at p, is h also continuous at p? If so, prove it. If not, give a counterexample and prove that it is a counterexample.

(b) (15 points) If both f and g are differentiable at p, is h also differentiable at p? If so, prove it. If not, give a counterexample and prove that it is a counterexample.

Problem 7 (50 points).

If $S \subset \mathbb{R}$ and $x \in \mathbb{R}$, we say x is an *accumulation point of* S if (and only if) for all $\epsilon > 0$, there exists some $y \in S$ with $y \neq x$ and $|x - y| < \epsilon$.

(a) (10 points) What are all accumulation points of the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$? (You need not prove your answer.)

(b) (10 points) What are all accumulation points of the set $S = (0, 1)$? (You need not prove your answer.)

(c) (15 points) Suppose that S is an infinite subset of the interval $[0, 1]$. Prove that there exists some $x \in \mathbb{R}$ such that x is an accumulation point of S.

(d) (15 points) Define the *closure* of S to be the set

$$\bar{S} = \{x \in \mathbb{R} \text{ such that either } x \in S \text{ or } x \text{ is an accumulation point of } S\}.$$

Prove that

$$\bar{\bar{S}} = \bar{S}$$

(in words, "the closure of the closure of S is the closure of S.")