Canonical quantization

Another viewpoint on $\mathcal{H}$: "canonical quantization."

Take $X = \mathbb{R}$, look at the space $\mathcal{S}$ of critical pts of S. ("classical solutions")

Let's describe $\mathcal{S}$ in our case. A tangent vector to $\mathcal{C}$ is a "small variation" $\phi(t) \to \phi(t) + \delta\phi(t)$

We compute the derivative of S in the direction of this tangent vector:

$$\delta S = \int dt \; \tfrac{1}{2}\delta\left(g_{jk}(\phi)\dot\phi^j\dot\phi^k\right) + \delta(V(\phi))$$

$$= \int dt \; \frac{1}{2}\frac{\partial g_{jk}(\phi)}{\partial\phi^i}\dot\phi^j\dot\phi^k\delta\phi^i + g_{ji}(\phi)\dot\phi^j\delta\dot\phi^i + \frac{\partial V(\phi)}{\partial\phi^i}\delta\phi^i$$

$$= \int dt \left[\frac{1}{2}\frac{\partial g_{jk}(\phi)}{\partial\phi^i}\dot\phi^j\dot\phi^k - \frac{\partial}{\partial t}\left[g_{ji}(\phi)\dot\phi^j\right] + \frac{\partial V(\phi)}{\partial\phi^i}\right]\delta\phi^i$$

$\left[g_{ij,k} = \frac{\partial}{\partial\phi^k}g_{ij}\right]$

This vanishes for all variations $\delta\phi^i$ $\iff$ $\dfrac{\partial}{\partial t}\left[g_{ji}(\phi)\dot\phi^j\right] = \dfrac{1}{2}g_{jk,i}\dot\phi^j\dot\phi^k + \dfrac{\partial V}{\partial\phi^i}$

$$g_{ji}\ddot\phi^j + \left[g_{ji,k} - \tfrac{1}{2}g_{jk,i}\right]\dot\phi^j\dot\phi^k = \frac{\partial V}{\partial\phi^i}$$

i.e. $\nabla_{\dot\phi}\dot\phi = g^{-1}(dV)$. So ϕ is parameterized geodesic (deformed by potential V)

(NB: you might have expected $-V$ here! Euclidean time reverses the sign)

By looking at initial data at any time t_0, we can identify $\mathcal{S} \simeq TM$.

Using metric in M, also $\mathcal{S} \simeq T^*M$.

There is a natural action of $\mathbb{R}$ on $\mathcal{C}$, under which S is invariant:

$$\phi(t) \longrightarrow \phi(t+c)$$

Generated by a vector field W: $\quad \delta\phi(t) = \varepsilon\dot\phi(t) \qquad \left(W = \int dt \; \dot\phi(t)\frac{\delta}{\delta\phi(t)}\right)$

This is called a local symmetry (since $\delta\phi(t)$ depends only on data at t).

It has $\delta L = \frac{d}{dt}L$.

Remarkable fact (Noether): local symmetries $\longleftrightarrow$ conservation laws.

e.g. let's consider a general time-independent local symmetry v.f. V, $\qquad \delta\phi = \varepsilon f(\phi,\dot\phi)$

under which we have $\delta L = c\frac{d}{dt}L$ for some $c \in \mathbb{R}$.

Let's evaluate δS for a variation of the form $\delta\phi = \varepsilon(t)f(\phi,\dot\phi) \qquad \delta\dot\phi = \dot\varepsilon f + \varepsilon\dot f$

We get $\delta S = \int dt\ \varepsilon(t)\left[\frac{\delta L}{\delta\phi}\cdot f + \frac{\delta L}{\delta\dot\phi}\cdot\dot f\right] + \dot\varepsilon(t)\frac{\delta L}{\delta\dot\phi} f$

$$= \int dt\ \varepsilon(t)\ c\frac{dL}{dt} + \dot\varepsilon(t)\frac{\delta L}{\delta\dot\phi} f$$

$$= \int dt\ \varepsilon(t)\frac{d}{dt}\left[cL - \frac{\delta L}{\delta\dot\phi} f\right]$$

Now, suppose we restrict to $\phi \in \mathcal{S}$. Then $\delta S = 0$ for <u>any</u> variation, particularly this one!

So, we get $\quad 0 = \int dt\ \varepsilon(t)\frac{d}{dt}\left[cL - \frac{\delta L}{\delta\dot\phi} f\right]$ for <u>all</u> $\varepsilon(t)$, i.e.

$$0 = \frac{d}{dt}\left[cL - \frac{\delta L}{\delta\dot\phi} f\right]$$

And finally this says that $Q = cL - \frac{\delta L}{\delta\dot\phi} f(\phi,\dot\phi)$ obeys $\frac{dQ}{dt} = 0$ when $\phi \in \mathcal{S}$.

Many examples in <u>nature</u>. (energy, momentum, angular momentum...)

The relation between Q and V can be made very direct: introduce a <u>symplectic</u> <u>structure</u> on $\mathcal{S}$ by $\omega(\delta_1,\delta_2) = \delta_1\left[\frac{\delta L}{\delta\dot\phi}\right]\delta_2\phi - \delta_1\phi\,\delta_2\left[\frac{\delta L}{\delta\dot\phi}\right]$

Then $\omega^{-1}(dQ) = V$.

I.e. $Q|_{\mathcal{S}}$ is a Hamiltonian generating the action of V on $\mathcal{S}$.

(<u>Pf</u>: Compute $\omega(V,W) = W(Q)$, for any W)

Apply this in our case: conserved charge generating <u>time translations</u>?

Here $\delta\phi^i = \varepsilon f^i$ with $f^i = \dot\phi^i$, and $c = 1$; <u>so</u>

$$Q = cL - \frac{\delta L}{\delta\dot\phi^i} f^i = \left(\tfrac{1}{2}\|\dot\phi\|^2 + V(\phi)\right) - \|\dot\phi\|^2 = -\tfrac{1}{2}\|\dot\phi\|^2 + V(\phi)$$

This is the (classical) <u>Hamiltonian</u>. (Unfamiliar <u>sign</u> b/c of Euclidean continuation...)

Now, given a symplectic manifold $\mathcal{S}$, the pb of quantization is to find a

Hilbert space $\mathcal{H}$, such that functions f on $\mathcal{S}$ $\rightsquigarrow$ operators $\mathcal{O}_f \in \mathrm{End}(\mathcal{H})$

with: $[\mathcal{O}_f, \mathcal{O}_g] = \mathcal{O}_{\{f,g\}}$ [at least to first order in some parameter] where the "Poisson bracket" $\{f,g\} = \omega^{-1}(df, dg)$

In our example: say $Y = \mathbb{R}^n$ w/ flat metric. Then $\mathcal{S} = T^*\mathbb{R}^n$

In loc. coords (x^i, p_i) on $\mathcal{S}$, we have

$$\{x^i, p_j\} = \delta^i_j$$
$$\{x^i, x^j\} = 0$$
$$\{p_i, p_j\} = 0$$

Then quantize: $\mathcal{H} = L^2(\mathbb{R}^n)$

$x^i \rightsquigarrow$ multiplication by x^i

$p_i \rightsquigarrow \dfrac{\partial}{\partial x^i}$

This is essentially the unique possible quantiz^n of these functions which is unitary in the sense that x^i is Hermitian, p_i skew-Hermitian (Stone-von Neumann)

Quantization of more general $f(x,p)$ should give more general differential operators, but, suffers from ordering ambiguities: e.g. $xp = px$, so should it quantize to $x\dfrac{\partial}{\partial x}$ or $\dfrac{\partial}{\partial x}x$? $\left[\text{"} x\dfrac{\partial}{\partial x} + 1 \text{"} \right]$

Thus only the highest-order part is indep. of these choices.

We can make some definite choice, e.g. "Weyl ordering."

(But we won't succeed in getting $[\mathcal{O}_f, \mathcal{O}_g] = \mathcal{O}_{\{f,g\}}$ for all f, g. At best, can get it up to quadratic in p's.)

Now say Y is curved. Then we can try to globalize the above to get a quantiz^n of Y.

Natural guess: $\mathcal{H} = L^2(Y)$, quantize functions that are homogeneous of degree k on cotangent directions into k^{th} order differential operators. At least for $k=1$, there is a canonical way to do this: a degree-1 homog f^n on T^*Y is just a vector field, which indeed is a 1^{st} order diff op. But at higher degrees we will run into ordering problems.

Our main interest here is in quantizing the function $H = -\frac{1}{2}\|p\|^2 + V$.
This is supposed to give the Hamiltonian operator.

A natural proposal is $H \rightsquigarrow \frac{1}{2}\Delta + V$ since the symbol of Δ is indeed $-\|p\|^2$
(This is the H we found corresponded to one natural def. of path-$\int$ measure...)

But, if Y is not flat, there are various possible choices for the quantiz^{n} — we could have equally naturally replaced Δ by $\Delta + \alpha R$ where R is the scalar curvature and $\alpha \in \mathbb{R}$.
From this point of view, it seems it would be difficult to fix H precisely...

Later, we'll consider a supersymmetric theory where this ambiguity won't occur.

A remark on rel^{n} between path $\int$ and operator formalisms
("where the noncommutativity comes from"):

Consider $Y = \mathbb{R}$, $V = 0$. $\mathcal{S} = T^*\mathbb{R}$ $[\mathcal{O}_x, \mathcal{O}_p] = 1$

Then the most naive dictionary would be

$$\langle y_2 | e^{-Ht_3} \mathcal{O}_p e^{-Ht_2} \mathcal{O}_x e^{-Ht_1} | y_1 \rangle = \int_{\mathcal{C}_{I_{t_1+t_2+t_3}}[y_1,y_2]} \mathcal{D}\phi \; \dot{\phi}(t_1+t_2)\, \phi(t_1)\, e^{-S(\phi)}$$

$$\langle y_2 | e^{-Ht_3} \mathcal{O}_x e^{-Ht_2} \mathcal{O}_p e^{-Ht_1} | y_1 \rangle = \int_{\mathcal{C}_{I_{t_1+t_2+t_3}}[y_1,y_2]} \mathcal{D}\phi \; \phi(t_1+t_2)\, \dot{\phi}(t_1)\, e^{-S(\phi)}$$

As we take $t_2 \to 0$, we might formally write this as $\int \mathcal{D}\phi \; \phi(t_1)\dot{\phi}(t_1)\, e^{-S(\phi)}$

But if we try to really define that in the discretization,
we'll see 2 possible definitions:

$t_1 + \Delta t$
t_1
$t_1 - \Delta t$

either replace $\dot{\phi}(t_1)$ by $\phi(t_1+\Delta t)-\phi(t_1)$

or by $\phi(t_1)-\phi(t_1-\Delta t)$

This leads to considering 2 <u>different</u> integrals: as $\Delta t \to 0$ they look like

$$\frac{1}{\Delta t}\int dy^1\, dy^2\, dy^3 \;\; (y^2-y^1)\,y^2\, e^{-F(y)} \qquad \text{or} \qquad \frac{1}{\Delta t}\int dy^1\, dy^2\, dy^3 \;\; (y^3-y^2)\,y^2\, e^{-F(y)}$$

with $F(y)=\left[(y^1-y^2)^2+(y^3-y^2)^2\right]/\Delta t$

By $\int$ by parts, see that these two indeed <u>differ</u> by $\int dy^1\, dy^2\, dy^3\; e^{-F(y)}$

The difference is nontrivial even in the limit $\Delta t \to 0$!

(A baby example of the general phenomenon of "anomaly", very broadly interpreted: some unexpected dependence on details of regularization)