The free boson in 2 dimensions

Up until now, all the objects we've discussed have been rigorously defined (or at least should be rigorously definable.)

Now we are going to press on to a setting where this really isn't true: $\dim X > 1$.

Ex X Riem 2-manifold, Y Riem manifold, $V: X \to Y$

$$\mathcal{C} = \mathrm{Map}(X,Y), \quad S: \mathcal{C} \to \mathbb{R}, \quad S[\varphi] = \int_X \tfrac{1}{2}\|d\varphi\|^2 + V(\varphi)$$

"2-dimensional σ model with target Y"

Formally we could imagine doing all the things we did in $\dim X \leq 1$.
Path integrals, observables, canonical quantization. But, one meets new difficulties.

Ex Suppose $X = S^1_L \times S^1_\beta$, $Y = \mathbb{R}$, $V = 0$.
Then can Fourier expand $\varphi: \underset{x}{S^1_L} \times \underset{t}{S^1_\beta} \to \mathbb{R}$ in modes

$$\varphi(x,t) = \sum_{n \in \mathbb{Z}} a_n(t) e^{2\pi i n x / L} \qquad \bar{a}_n = a_{-n}$$

$$S[\varphi] = \int dt \ \tfrac{1}{2}\dot{a}_0(t)^2 + \sum_{n>0}\left[|\dot{a}_n(t)|^2 + \frac{4\pi^2 n^2}{L^2} |a_n(t)|^2 \right]$$

We've now "reduced" to 1 dimension but with an infinite set of fields.
So this is like an infinite set of decoupled (complex)
harmonic oscillators with frequencies $\omega_n = \frac{2\pi n}{L}$, plus one extra map $a_0: S^1_\beta \to \mathbb{R}$

[Decomposing each a_n into its real and imag parts, see that each complex h.o. is equivalent to two real ones.]

Now, what should we get for (say) $Z_{S^1 \times S^1}$?

Operator formalism would suggest $Z = \mathrm{Tr}_{\mathcal{H}} e^{-\beta H}$

where $\mathcal{H} = \left(\bigotimes_{n=1}^{\infty} \mathcal{H}_n \right) \otimes \mathcal{H}_0$ $\quad$ $\mathcal{H}_n$ = Hilbert space for h.o. w/ $\omega = \frac{2\pi n}{L}$

$\mathcal{H}_0$ = Hilb. sp. for the "zero mode" a_0

$$H = \sum_{n=1}^{\infty} \left(1 \otimes \cdots \otimes 1 \otimes H_n \otimes 1 \cdots \right) + H_0 \otimes 1 \otimes 1 \otimes \cdots$$

What are the eigenvalues of H? Let's start with the lowest one.

Lowest eigenvalue E would naively come from summing all the lowest eigenvalues of H_n.

If we use our usual quantization $H_n = -\frac{1}{2} \frac{\partial^2}{\partial a_n^2} + \frac{1}{2} \omega_n^2 a_n^2$ (and say $H_0 = 0$)

that would give $E = \sum E_n = 2 \sum_{n>0} \frac{\omega_n}{2} = \frac{2\pi}{L} \sum_{n>0} n$, evidently divergent!

That would be a disaster.

But, as we mentioned, the rules of canonical quantiz^n allow a shift $H \to H + c$

What shift could be produced "naturally" by the path integral?

Regulate: replace $E_n \to E_n f\left(\frac{n}{L}\right)$ where f is some cutoff function.

e.g. $f\left(\frac{n}{L}\right) = e^{-\varepsilon \frac{n}{L}}$, get $E(\varepsilon) = \frac{2\pi}{L} \sum_{n>0} n e^{-\varepsilon \frac{n}{L}}$

Then expand around $\varepsilon = 0$: $\quad E(\varepsilon) = \frac{2\pi L}{\varepsilon^2} - \frac{\pi}{6L} + \cdots$

The singularity is proportional to L! It can thus be removed by adding a term $\frac{2\pi}{\varepsilon^2}$ to the action ("local counterterm"). To take the limit $\varepsilon \to 0$ this term must be added.

After so doing, we get $\quad E = -\frac{\pi}{6L}$ $\quad\quad$ (" $\sum_n n = -\frac{1}{12}$ ")

(This answer doesn't depend on the function f we picked. Can also be obtained by the black magic of zeta-function regularization.)

The rest of the spectrum of H is determined as usual. One fine point is the contribution from $\mathcal{H}_0$ — regulate this by replacing $L^2(\mathbb{R}) \rightsquigarrow L^2(S^1_V)$ to make the spectrum discrete. As $V\to\infty$, leading behavior of $\mathrm{Tr}_{\mathcal{H}_0} e^{-\beta H}$ is $V/2\pi\sqrt{\beta/L}$.

Altogether we get

$$Z_{S^1_\beta \times S^1_L} = \frac{V}{2\pi\sqrt{\frac{\beta}{L}}}\,\eta(q)^{-2} \qquad \text{with} \qquad q = e^{-2\pi\beta/L}, \quad \eta(q) = q^{1/24}\prod_{n=1}^{\infty} 1-q^n$$

Modular properties of $\eta(q)$ $\left(q = e^{2\pi i\tau},\ i\tau = \frac{\beta}{L}\right)$ imply that Z is invariant under $L \leftrightarrow \beta$, as the path integral perspective would predict! (This depends on the $q^{\frac{1}{24}}\ldots$)

Generalizes to a tilted torus:

$$Z_{T^2_\tau} = \frac{V}{2\pi\sqrt{\mathrm{Im}\,\tau}}\,|\eta(\tau)|^{-2}$$

inv^t under $\tau \to \tau+1$, $\tau \to -\frac{1}{\tau}$

An interesting new phenomenon emerges when we consider the theory with $Y = S^1_{2\pi R}$ and $X = T^2$:

- in path $\int$, the config. space $\mathcal{C}$ is disconnected, $\mathcal{C} = \bigsqcup_{(n_1,n_2)\in\mathbb{Z}^2} \mathcal{C}_{n_1,n_2}$

- in operator formalism, $\mathcal{A} = \bigsqcup_\ell \mathcal{A}_\ell$, and each $\mathcal{A}_\ell$ has a factor S^1, which quantizes to $L^2(S^1)$ hence to $\bigoplus_{m\in\mathbb{Z}}$

$$\Longrightarrow \quad \mathcal{H} = \bigoplus_{(n_1,m)\in\mathbb{Z}^2} \mathcal{H}_{(n_1,m)}$$

[Warning, $m \leftrightarrow n_2$ are morally dual to one another, not equal!!]

Lowest eigenvalue of H in each sector: $E_{n_1,m} = \left(\frac{m}{R}\right)^2 + (n_1 R)^2$

One obtains

$$Z_{T^2} = \frac{1}{|\eta(\tau)|^2}\sum_{(n_1,m)} q^{\frac{1}{4}\left(m/R - n_1 R\right)^2}\,\bar{q}^{\frac{1}{4}\left(m/R + n_1 R\right)^2}$$

Invariant under $R \leftrightarrow \frac{1}{R}$! This is the first manifestation of "T-duality" — the basis of mirror symmetry...

Why is T-duality true?

Path integral derivation: we'll consider an enhanced action S_{big} such that integrating over some variables in S_{big} leads either to $S(R)$ or to $S(\frac{1}{R})$

Namely: $\mathcal{C} = \left\{ \begin{matrix} \varphi : X \to S^1_{2\pi} \\ B \in \Omega^1(X) \end{matrix} \right\}$ X any Riemann surface

$$S_{big} = \frac{1}{2\pi} \int_X \frac{1}{2R^2} \|B\|^2 + \frac{i}{2\pi} \int B \wedge d\varphi$$

$$Z = \int_{\mathcal{C}} \mathcal{D}\varphi \, \mathcal{D}B \, e^{-S_{big}}$$

One possibility: eliminate B first.

$$S_{big} = \frac{1}{2\pi} \int_X \frac{1}{2R^2} \|B - iR^2 \star d\varphi\|^2 + \frac{R^2}{4\pi} \int_X \|d\varphi\|^2$$

and Gaussian $\int$ over B then just gives some determinant depending on X but totally decoupled from φ. Thus reduce to "effective" theory of φ with

$$S = \frac{R^2}{4\pi} \int \|d\varphi\|^2.$$

Other possibility: integrate over φ first. φ appears only linearly in S_{big}.

Recall that $\int_{\mathbb{R}} dx \, e^{ixy} = \delta(y)$, $\sum_{n \in \mathbb{Z}} e^{2\pi i n a} = \sum_{m \in \mathbb{Z}} \delta(a-m)$

Here integrating over φ will produce a constraint on B.

Expand $d\varphi = d\varphi_0 + \sum_{i=1}^{2g} 2\pi n_i \omega^i$

ω^i representatives for $H^1(\Sigma, \mathbb{Z})$

$\varphi_0 : \Sigma \to \mathbb{R}$

$\{n_i\} \in \mathbb{Z}^{2g}$ label topology of φ

Integration over φ means int. over φ_0 and sum over n_i.

Integral over φ_0 imposes the constraint $dB = 0$. Every closed 1-form can be written as

$$\int B \wedge d\varphi = \int B \wedge \left(d\varphi_0 + \sum' 2\pi n_i \omega^i\right)$$

$$B = d\vartheta_0 + 2\pi \sum_{i=1}^{2g}{}' a^i \omega_i \qquad a^i \in \mathbb{R} \qquad \int_X \omega^i \wedge \omega_j = \delta^i_j$$

Plug this in: effective action becomes

$$S(\vartheta_0, a^i) = \frac{1}{4\pi R^2}\int_X \|B\|^2 + i\int_X \left(d\vartheta_0 + \sum_i{}' a^i \omega_i\right) \wedge \sum_j{}' 2\pi n_j \omega^j$$

$$= \frac{1}{4\pi R^2}\int_X \|B\|^2 + 2\pi i \int_X a^i n_j \left(\omega_j \wedge \omega^i\right)$$

$$= \frac{1}{4\pi R^2}\int_X \|B\|^2 + 2\pi i \int_X a^i n_i$$

Now use $\displaystyle\sum_{n\in\mathbb{Z}}{}' e^{2\pi i n a} = \sum_m{}' \delta(a-m)$

So the $\sum$ over n_i imposes the constraint that all $a^i \in \mathbb{Z}$!

Altogether our constraints say that $B = d\vartheta$ where $\vartheta : \Sigma \to S^1_{2\pi}$
(ϑ not unique, but that's OK)
So, we have reduced to an <u>effective action</u>

$$S(\vartheta) = \frac{1}{4\pi R^2}\int_X \|d\vartheta\|^2$$

A bizarre-looking equivalence between $\int$ over maps $\Sigma \to S^1_{2\pi R}$ and $\Sigma \to S^1_{\frac{2\pi}{R}}$!

Moreover it can be extended — not only to partition functions but also to observables in the two theories.

For example: $\forall x \in X$, have a map "$d\varphi(x)$": $\mathcal{C} \to T^*_x X$
given by $\varphi \mapsto d\varphi(x)$

This gives a ($T^*_x X$-valued) observable.
Similarly can define observable $\star d\varphi(x)$.

T-duality says $\langle d\varphi(x_1)\, d\varphi(x_2) \rangle$ in theory w/ $Y = S^1_R$
$$\langle d\varphi(x_1)\, d\varphi(x_2) \rangle = \langle \star d\varphi(x_1) \star d\varphi(x_2) \rangle$$
" " " $Y = S^1_{1/R}$

(both sides valued in $T^*_{x_1} X \otimes T^*_{x_2} X$)

And similarly for arbitrary insertions like $\langle d\varphi(x_1)\, d\varphi(x_2) \star d\varphi(x_3) \star d\varphi(x_4) \rangle, \ldots$

("Provable" by a generaliz^n of what we did above — make an appropriate insertion in the path int. of the theory w/ action S_{big}, see what it becomes on both sides...)

In short: T-duality maps $d\varphi \longleftrightarrow \star d\varphi$

We thus have a QFT s.t. all quantities of interest can be computed in 2 distinct ways. "One theory, 2 different classical descriptions"

Next, we'll see an example where we have infinitely many diff. classical descrip...

Symmetries

In general both $\mathrm{Isom}(X)$ and $\mathrm{Isom}(Y)$ act on $\mathcal{C}$ preserving S.

e.g. if $X = \mathbb{R}^2$ then $G = \mathrm{Isom}(\mathbb{R}^2) = IO(2)$ acts ("Poincaré group").

$\mathfrak{g}$ spanned by: $\left.\begin{array}{ll} H & \text{corresponding to translation in } \underline{\text{time}} \\ P & \quad \text{"} \qquad \text{"} \qquad \text{"} \qquad \text{"} \quad \underline{\text{space}} \\ J & \quad \text{"} \qquad \text{"} \quad \underline{\text{rotation}} \end{array}\right\} \rightsquigarrow$ 3 odd vector fields on $\mathcal{C}$ $\circlearrowleft$

The corresp. Noether charges are <u>energy</u>, <u>momentum</u>, <u>angular momentum</u>.

SUSY

As in 1d, we can augment $\mathfrak{g}$ by some <u>odd</u> vector fields. $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$

$\mathfrak{g}^1$ is then a repⁿ of $\mathfrak{g}^0$. Consider <u>flat</u> X, so $\mathfrak{g}^0 = ISO(d)$.

In all cases we'll consider, $\mathfrak{g}^1_{\mathbb{C}}$ is $\oplus$ of copies of <u>spinor</u> representations $S^\pm$ of $ISO(d)$.

("Spin-statistics thm" says this is what always happens in QFT.)

Thus label distinct amounts of SUSY by pair of multiplicities $\mathcal{N} = (n^+, n^-)$.

The σ-model admits various supersymmetric extensions. Most famous one: if Y Kähler we can define $\mathcal{N} = (2,2)$ SUSY σ-model w/ target Y.

Even more interesting, for non-flat X, and Y Kähler, we can <u>still</u> define an extension of the σ-model which has 1 odd symmetry (and no even ones). In fact we can do it in 2 distinct ways:

<u>A model</u> (localizes on holomorphic maps)

<u>B model</u> (localizes on constant maps)

[Witten: "Topological σ models"]

T-duality continues to exist in this setting.

This route would lead eventually to <u>mirror symmetry</u>...