BPS Monopoles

Now, let's see why $\mathcal{I}(\mathcal{H}_\gamma(u)) \neq 0$ for $\gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ electric / magnetic by producing the relevant states.

Their description will be very different from the states we produced using the field operators: get them by directly exhibiting a "piece" of the space $\mathcal{S}$ of classical solutions, whose quantiz^n will yield the states we seek.

To find it, first look for sol^n which is time-independent, and fixed point of some of the odd symmetries: 't Hooft-Polyakov monopole.

Boundary condition at ∞: $\phi \sim \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$

Expand adjoint fields on ON basis of $su(2)$. Introduce $m = ag$ (Higgs scale). Then the solution with $A_0 = 0$ is

$$A_i^b = \varepsilon_{bij} \frac{x^j}{gr^2}\left(1 - \frac{mr}{\sinh mr}\right) \longrightarrow \varepsilon_{bij} \frac{x^j}{gr^2} \quad \text{as } r \to \infty \qquad \left(r^2 = \textstyle\sum_i x^{i2}\right)$$

$$\phi^b = \delta_i^b \frac{x^i}{gr^2}\left(mr \coth(mr) - 1\right) \longrightarrow \delta_i^b\, a \frac{x^i}{r} \quad \text{as } r \to \infty$$

It has classical energy-momentum $p = \begin{pmatrix} \frac{4\pi a}{g^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$

i.e. $M = \frac{4\pi a}{g^2}$ Very heavy as $g \to 0$; a "non-perturbative" object.

This config. is a solution of Bogomolny equations $F = \star D\phi$ (after reduction to 3 dim)

$\Longrightarrow$ it is annihilated by ½ of the odd vector fields!

[These eq. are known to be the dimensional reduction of SDYM equation $F^+=0$ from 4 to 3 dimensions. However, that is not how they arose here — we are working on $\mathbb{R}^{3,1}$ not $\mathbb{R}^4$. We could view them as arising via a different dimensional reduction: begin with a 6d theory in $\mathbb{R}^{5,1}$ and then look at configs. which are invariant under shift along $\mathbb{R}^{2,1}$. Factorizing this into first $\mathbb{R}^{1,1}$ and then another $\mathbb{R}^1$, we recover the statement about dim. red. of SDYM.]

Now, what does this configuration look like, from the point of view of the low energy physics?

At long distance ($|\vec{x}| \to \infty$) it has approximately $F=0$, $D\phi=0$, but $\phi \neq 0$

$\Longrightarrow$ looks reducible, $E = \mathcal{L} \oplus \mathcal{L}^{-1}$. This $\mathcal{L}$ tells us how the object looks in the effective $U(1)$ theory. Using our explicit expression for the solution we get that $\mathcal{L}$ is topologically nontrivial over S^2 given by $|\vec{x}| = R$, $t = \text{fixed}$.

This is just what one expects for the field of a monopole: $\oint_{S^2} F \neq 0$

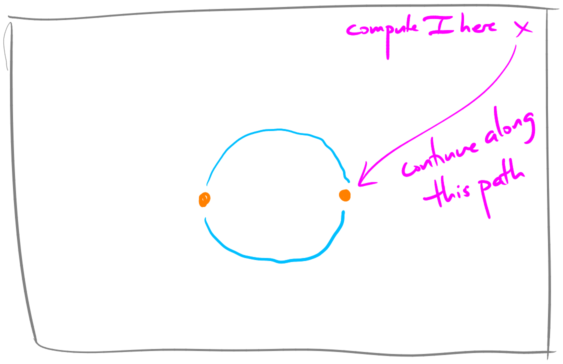
So this particle is carrying magnetic charge from the POV of the low energy $U(1)$ theory!

Now — how does this classical construction influence the quantum theory?

This is not so easy to understand, but the heuristic idea is that quantization of the moduli space of these monopole solutions — sitting inside the full space $\mathcal{S}$ of classical solutions — produces states in $\mathcal{H}^1_\gamma(u)$. Moreover the fact that the classical sol^n's are annihilated by odd vector fields goes over to the stmt that the corresp. states are in short representations of $\underset{\sim}{\mathfrak{g}}$.

This gives the desired result $\mathcal{I}(\mathcal{H}^1_\gamma(u)) \neq 0$.

Warning: the deformation invariance of $\mathcal{I}(\mathcal{H}^1_\gamma(u))$ can be violated when u meets a codim-1 "wall", where the continuum of multi-particle states touches $\mathcal{H}^1_\gamma(u)$. These "walls" fortunately can be studied & in the theory we are considering they don't affect our conclusion:



$$\mathcal{I}(\mathcal{H}^1_\gamma(u)) \neq 0 \quad \text{as } u \longrightarrow \Lambda$$