Seiberg-Witten equations

Now reconsider the topologically twisted version of $\mathcal{N}=2$ super Yang-Mills.
We've already seen that this theory computes Donaldson invariants.

$$\langle \mathcal{O}^{(0)} \mathcal{O}^{(2)}(\gamma_1) \cdots \mathcal{O}^{(2)}(\gamma_k) \rangle$$

The correlation fncs are independent of the metric g on X.
So, consider replacing $g \to tg$ and taking $t \to \infty$.
In this limit we should be able to compute using the effective action and get the exact answer!

Naively we might expect that effective theory on X could be obtained just by twisting the effective theory we had on $\mathbb{R}^4$. Not obvious but seems to be almost OK in this case.
There can be additional terms (which vanish on $\mathbb{R}^4$). These extra terms are of the form

$$\int_X A(u)\, \mathrm{Tr}(R \wedge * R) + B(u)\, \mathrm{Tr}(R \wedge R) + C(u)\, F \wedge w_2(X)$$

(more general R dependence forbidden since we must get topological invariants)

$\Longrightarrow$ Need to determine $A(u), B(u), C(u)$ somehow. (Witten originally did it just by comparing to known facts about D – SW relation; later studied more systematically...)

The observables $\mathcal{O}^{(0)}, \mathcal{O}^{(2)}(\gamma)$ should also map to some observables of the effective theory.

$\mathcal{O}^{(0)} = \mathrm{Tr}(\varphi^2) \rightsquigarrow \mathcal{O}^{(0)}_{\text{eff}} = u$ (almost by definition)

$\mathcal{O}^{(2)}(\gamma)$ obtained from $\mathcal{O}^{(0)}$ by descent (using SUSY of the original theory) $\rightsquigarrow$ build $\mathcal{O}^{(2)}_{\text{eff}}$ from $\mathcal{O}^{(0)}_{\text{eff}}$ the same way (using SUSY of the eff. theory)

Contact terms can appear at the $\cap$ between γ_i $\Longrightarrow$ one more undetermined $T(u)$.

Modulo these undet. f^n's, we've now specified what to compute.

But how do we compute? [Moore-Witten]

Basic structure: $Z = Z_{u\text{-plane}} + Z_{u=\Lambda} + Z_{u=-\Lambda}$

To get $Z_{u\text{-plane}}$, write the effective action, note that the χ field has b_2^+ zero modes.

If $b_2^+ > 1$, no way to absorb these and get answer scaling as t^0 $\Rightarrow$ $Z_{u\text{-plane}} = 0$

For $Z_{u=\Lambda}$, write a different effective action, compute by localization.

$\bar{2} \otimes 2 \to 3$

The relevant moduli space: [for $\lambda \in H^2(X,\mathbb{Z}) + \frac{1}{2} w_2(X)$]

$$\mathcal{M}_\lambda = \left\{ \begin{matrix} \nabla \ U(1) \text{ conn in line bundle } L, \ \frac{c_1(L)}{2} = \lambda \\ M \in \Gamma(S^+ \otimes L^{1/2}) \end{matrix} : \begin{matrix} F^+ = (\bar{M}M)^+ \\ \cancel{D}M = 0 \end{matrix} \right\} \Big/ \sim$$

["Spin^c structure": S^+ and $L^{1/2}$ need not separately exist but this $\otimes$ does]

$$\dim \mathcal{M}_\lambda = \lambda^2 - \frac{2\chi + 3\sigma}{4}$$

NB: unlike in Donaldson theory this moduli space is compact!

As in Donaldson theory we now map observables of the low energy theory to closed differential forms on $\mathcal{M}_\lambda$. Thus define Seiberg-Witten invariants as appropriate integrals over $\mathcal{M}_\lambda$.

The formulas relating Donaldson to SW inv'ts are in general pretty complicated, even if $b_2^+ > 1$.

But, there's a nice special case: say X is of SW simple type if the only nonvanishing correlation func. $\langle \cdots \rangle_\lambda$ arise for $\dim \mathcal{M}_\lambda = 0$.

In this case define $SW(\lambda) = \langle 1 \rangle_\lambda$ (counts points of $\mathcal{M}_\lambda$ w/ signs)

Now, for $S \in H_2(X,\mathbb{Z})$ and $w \in H^2(X,\mathbb{Z})$ define <u>Donaldson generating function</u>

$$Z^{w}_{DW}(S) = \left\langle e^{p\mathcal{O}^{(0)} + \mathcal{O}^{(2)}(S)} \right\rangle_w$$

fixed Stiefel-Whitney class for SO(3) bundles: dependence is <u>almost</u> only on w mod 2 but the lift w enters <u>orientation</u>

A very naive guess for the relation would be something like

$$Z_{DW} \sim \sum_{\lambda}{}' \left(e^{p\lambda} + e^{-p\lambda}\right) SW(\lambda)$$

Correct version is given by <u>Witten's magic formula</u>:

$$Z^{w}_{DW}(S) = 2^{1+\frac{7}{4}\chi+\frac{11}{4}\sigma} \sum_{\lambda} e^{2\pi i\left(\frac{w}{2}\cdot\lambda+\frac{w^2}{4}\right)} \times \Big[\underbrace{e^{2p+\frac{1}{2}S^2}\, e^{2(S,\lambda)}}_{\text{from } u=2} + \underbrace{i^{\frac{\chi+\sigma}{4}-w^2}\, e^{-2p-\frac{1}{2}S^2}\, e^{-2i(S,\lambda)}}_{\text{from } u=-2} \Big] SW(\lambda)$$

basically "e^{2pu}" at $u=+1$

contact term

R-symmetry anomaly

In more general situations there are also formulas relating Donaldson to SW but they are much more involved!