Complex Geometry: Exercise Set 3

Exercise 1

Complete the proof of the equivalence of five characterizations of integrability which we stated in class, by showing that if $\bar{\partial}^2 = 0$ then the distribution $T^{0,1}X \subset T_{\mathbb{C}}X$ is closed under Lie bracket.

Exercise 2

1. Suppose given an oriented real surface M with a Riemannian metric g. Define a canonical complex structure I_g on M. Your construction should be functorial in the sense that an orientation-preserving isometry $(M, g) \to (M', g')$ gives a holomorphic map $(M, I_g) \to (M', I'_g)$. Moreover, for any $f : M \to \mathbb{R}$ you should have $I_g = I_{e^f g}$. (Hence I_g actually depends only on the *conformal structure* induced by the metric g.)
2. Show conversely that if $I_g = I_{g'}$ then $g = e^f g'$ for some f.
3. Show that every complex structure on M is obtained as I_g for some g. (Thus on a real surface, *complex structures* and *conformal structures* are equivalent.)

Exercise 3

In lecture we defined a holomorphic line bundle $\mathcal{L}_\alpha$ over the torus Σ_τ, for any $\alpha \in \mathbb{C}$.

1. Show that $\mathcal{L}_\alpha \otimes \mathcal{L}_\beta \simeq \mathcal{L}_{\alpha+\beta}$.
2. Show that $\mathcal{L}_\alpha^* \simeq \mathcal{L}_{-\alpha}$.

Exercise 4

Let $U \subset \mathbb{C}^n$ be some open set. Consider the topologically trivial C^∞ complex vector bundle $V = U \times \mathbb{C}^r$ over U. We stated in class that a $\bar{\partial}$ operator on sections of V, obeying Leibniz rule, is equivalent to a holomorphic structure on V. Suppose given two such operators $\bar{\partial}^{(1)}$, $\bar{\partial}^{(2)}$. By the above we obtain two holomorphic vector bundles E_1, E_2. Show that $E_1 \simeq E_2$ if and only if there exists a map $g : U \to GL(r, \mathbb{C})$ such that for all C^∞ sections of V over U we have

$$\bar{\partial}^{(1)} s - \bar{\partial}^{(2)} s = (g^{-1} \bar{\partial} g) s.$$

Exercise 5

Consider a compact complex curve X. Define a meromorphic 1-form ω on X to be one which in local coordinates is $\omega = f(z)\, dz$ with $f(z)$ meromorphic.

1. For any $p \in X$ define the *residue* $\mathrm{Res}_p\, \omega$ of a meromorphic 1-form ω. Show in particular that it does not depend on the choice of local coordinate around p. (In contrast, there is no good invariant notion of the residue of a meromorphic *function*!)
2. Prove that $\sum_{p \in X} \mathrm{Res}_p\, \omega = 0$.

Exercise 6

Say X is a complex manifold with a submanifold Y. We call Y a *complex submanifold* if there is a holomorphic atlas of X which when restricted to Y gives a holomorphic atlas of Y. Show that if Y is a complex submanifold then $TY \subset TX$ is closed under the almost complex structure operator I of X. (The converse is also true.)

Exercise 7

(These are easy — the point of putting them here is just that they are statements one should keep in RAM.)

1. Let $f : U \to V$ be a holomorphic map. Show that pullback f^* preserves bidegree of complexified differential forms, i.e. takes $\Omega^{p,q}(V) \to \Omega^{p,q}(U)$.
2. Show that if $\alpha \in \Omega^{*,*}(U)$ is *real* ($\alpha = \bar{\alpha}$) and concentrated in a single bidegree, then $\alpha \in \Omega^{p,p}(U)$.
3. Show that $\overline{\partial\alpha} = \bar{\partial}\bar{\alpha}$.