Holomorphic tangent bundle

Consider $X = \mathbb{C}^n$. Write $z_i = x_i + iy_i$. $T(\mathbb{C}^n)$ is real of dim $= 2n$.

Let $\dfrac{\partial}{\partial z_i} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial x_i} - i\dfrac{\partial}{\partial y_i}\right] \in (T\mathbb{C}^n)_{\mathbb{C}} = T_{\mathbb{C}}\mathbb{C}^n$, similarly $\dfrac{\partial}{\partial \bar z_i}$

Now define an almost-$\mathbb{C}$ structure on $T\mathbb{C}^n$ by using $T\mathbb{C}^n \simeq \mathbb{C}^n$, or explicitly

$$I\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \qquad \text{i.e.} \qquad I = \begin{cases} +i & \text{on } T^{1,0}\mathbb{C}^n = \operatorname{Span}\left\{\dfrac{\partial}{\partial z_i}\right\} \\ -i & \text{on } T^{0,1}\mathbb{C}^n = \operatorname{Span}\left\{\dfrac{\partial}{\partial \bar z_i}\right\} \end{cases}$$

$$I\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$$

Prop If $f: \mathbb{C}^n \to \mathbb{C}^n$ then in the basis $\left\{\dfrac{\partial}{\partial z_1}, \dots, \dfrac{\partial}{\partial z_n}, \dfrac{\partial}{\partial \bar z_1}, \dots, \dfrac{\partial}{\partial \bar z_n}\right\}$ for $T_{\mathbb{C}}\mathbb{C}^n$, df is represented by $\begin{bmatrix} J & K \\ \bar K & \bar J \end{bmatrix}$ where $J = \left[\dfrac{\partial f_i}{\partial z_j}\right]_{i,j}$, $K = \left[\dfrac{\partial f_i}{\partial \bar z_j}\right]_{i,j}$.

Pf The dual basis is $\{dz_1, \dots, dz_n, d\bar z_1, \dots, d\bar z_n\}$.

$$dz_i\left(df\left(\frac{\partial}{\partial z_j}\right)\right) = \frac{\partial f_i}{\partial z_j}, \text{ etc.} \qquad \blacksquare$$

Cor If $f: \mathbb{C}^n \to \mathbb{C}^n$ hol then df is represented by $\begin{bmatrix} J & 0 \\ 0 & \bar J \end{bmatrix}$. In pt, $df(Iv) = I(df(v))$.

Prop If X a $\mathbb{C}$ mfd, TX carries natural structure of holomorphic vector bundle.

Pf Trivialized over the patches U_α of a hol atlas.

Get a complex structure I_x on TX by $I_x = \psi_\alpha^{-1} \circ I_{\varphi_\alpha(x)} \circ \psi_\alpha$

(this is independent of α, since $\psi_\beta \circ \psi_\alpha^{-1} = d(\varphi_\beta \circ \varphi_\alpha^{-1})$ and hence commutes with I, giving $\psi_\alpha^{-1} \circ I_{\varphi_\alpha(x)} \circ \psi_\alpha = \psi_\beta^{-1} \circ I_{\varphi_\beta(x)} \circ \psi_\beta$)

Thus get decomposition $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$.

Moreover, transition functions are $\psi_{\alpha\beta} = \begin{bmatrix} J_{\alpha\beta} & 0 \\ 0 & \bar J_{\alpha\beta} \end{bmatrix}$ where J is Jacobian of $\varphi_\alpha \circ \varphi_\beta^{-1}$.

$J_{\alpha\beta}$ is holomorphic $\Rightarrow T^{1,0}X$ has natural holomorphic structure.
Underlying real bundle is TX. $\blacksquare$

Def An almost $\mathbb{C}$ manifold is a pair (X, I) where X is a real manifold and I is a C^∞ section of $\mathrm{Hom}(TX, TX)$ with $I^2 = -\mathbb{1}$.

Cor A complex manifold is almost complex.