Sheaves

As we've seen, global hol. functions on compact complex X aren't too interesting. Nevertheless, one can build interesting global invariants from purely holomorphic stuff. The right language for that: sheaf cohomology.

M any topological space.

Def A presheaf $\mathcal{F}$ of abelian groups on M is:

- an abelian group $\mathcal{F}(U)$ for $U \subset M$ open
- a hom. $r_{U,V}: \mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subset U$

(could also replace groups $\to$ rings, sets, etc...)

("restriction")

such that $r_{U,U} = \mathbb{1}$ and if $W \subset V \subset U$ then $r_{V,W} \circ r_{U,V} = r_{U,W}$.

Def Call $\mathcal{F}$ a sheaf if also: for any $\{U_i\}$, w/ $U = \bigcup U_i$,

- $\left[f, g \in \mathcal{F}(U) \text{ and } r_{U,U_i}(f) = r_{U,U_i}(g) \ \forall i\right] \Rightarrow f = g$
- $\left[f_i \in \mathcal{F}(U_i) \text{ and } r_{U_i, U_i \cap U_j}(f_i) = r_{U_j, U_i \cap U_j}(f_j)\right] \Rightarrow \exists f \in \mathcal{F}(U)$ w/ $r_{U,U_i}(f) = f_i$

Ex

- $\mathcal{F}(U) = C^\infty(U)$, w/ r = restriction, is a sheaf (of rings). Call it "C^∞".
- $\mathcal{F}(U) = \mathbb{Z}$, w/ $r = \mathrm{id}$, is a presheaf, but not a sheaf if M contains 2 disjoint open sets.
- $\mathcal{F}(U) = \{$locally constant $\mathbb{Z}$-valued functions$\}$ is a sheaf.
  (Call this sheaf $\underline{\mathbb{Z}}$ (or sometimes just $\mathbb{Z}$); similarly $\underline{\mathbb{R}}$, $\underline{\mathbb{C}}$, $\underline{G}$)
- If V is a vector bundle over M, $\mathcal{F}(U) = C^\infty$ sections of $V|_U$ is a sheaf.
- "Skyscraper sheaf" S_x for any $x \in M$: $S_x(U) = \begin{cases} 0 & \text{if } x \notin U \\ \mathbb{C} & \text{if } x \in U \end{cases}$

If $M = X$ complex, even more:

- Let $\mathcal{O}$ be the sheaf of hol functions. (additive gp)
- $\mathcal{O}^\times$ " " " " nonvanishing hol functions. (multiplicative)
- Also can define meromorphic function to be one which looks locally like f/g f, g hol.
  Let $K^\times$ be sheaf of meromorphic functions. (multiplicative)
- Similarly define sheaves $\Omega^{p,q}$, $\Omega^{p,q}(E)$, Ω^p_{hol}, ...

If E hol v.b. over X let E also denote its sheaf of hol sections.

Def $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules if:

- $\mathcal{F}(U)$ is a module over $\mathcal{O}(U)$
- $\mathcal{O}(U) \circlearrowright \mathcal{F}(U)$ commutes, ie $r_{U,V}(fs) = r_{U,V}(f)\, r_{U,V}(s)$

$$\begin{tikzcd} \mathcal{O}(U) \arrow[d] & \circlearrowright & \mathcal{F}(U) \arrow[d] \\ \mathcal{O}(V) & \circlearrowright & \mathcal{F}(V) \end{tikzcd}$$

$\mathcal{F}$ is locally free of rank r if in addition there's a covering $\{U_i\}$ of X s.t. $\mathcal{F}|_{U_i} \simeq \mathcal{O}^{\oplus r}|_{U_i}$ as $\mathcal{O}|_{U_i}$-module.

Prop $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules: $\mathcal{F}$ is sheaf of sections of a hol v.b. of rank r
$\iff$ $\mathcal{F}$ is locally free $\mathcal{O}$-module of rank r.

Ex Define $\mathcal{O}(-[p])$ for $p \in X$.

Def A homomorphism of presheaves $\varphi: \mathcal{F} \to \mathcal{G}$ is a homomorphism $\varphi_U: \mathcal{F}(U) \to \mathcal{G}(U)$ $\forall U$

s.t.
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "r"] & \mathcal{G}(U) \arrow[d, "r"] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$
commutes $\forall V \subset U$.

Def/Prop Suppose $\mathcal{F}, \mathcal{G}$ are sheaves and $\varphi: \mathcal{F} \to \mathcal{G}$. Let $\operatorname{Ker}\varphi$ be the presheaf $(\operatorname{Ker}\varphi)(U) = \operatorname{Ker}(\varphi_U)$. Then $\operatorname{Ker}\varphi$ is a sheaf.

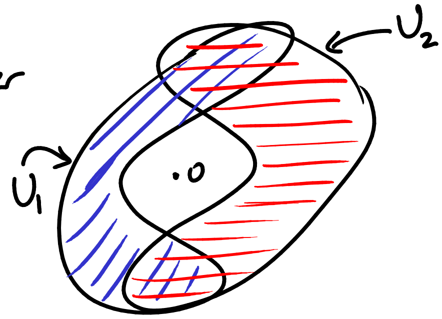
Pf Exercise.

Def φ is injective if $\operatorname{Ker}\varphi = 0$.

But $U \mapsto \operatorname{Im}\varphi_U$
$U \mapsto \operatorname{Coker}\varphi_U$
are only presheaves.

Ex $X = \mathbb{C}$, $\varphi : \mathcal{O}^\times \to \mathcal{O}^\times$
$\varphi(f) = f^2$

Then, consider



$z \in \operatorname{Im} \varphi_{U_1}$

$z \in \operatorname{Im} \varphi_{U_2}$

but $z \notin \operatorname{Im} \varphi_{U_1 \cap U_2}$

So $U \mapsto \operatorname{Im} \varphi_U$ is not a sheaf. (local doesn't glue to global)
Also $U \mapsto \operatorname{Coker} \varphi_U$ " " " " (local doesn't determine global)

In some sense φ is "locally" surjective (on "small enough open sets") but not globally... Let's make this precise.

Def Say $\mathcal{F}$ is a presheaf. The stalk of $\mathcal{F}$ at $x \in M$ is $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$.

i.e. $\mathcal{F}_x = \{ s \in \mathcal{F}(U) \text{ for some } U \ni x \} / \sim$

where $s_1 \sim s_2$ if $s_1 \in \mathcal{F}(U_1)$, $s_2 \in \mathcal{F}(U_2)$, $\exists\, U \subset U_1 \cap U_2$ s.t. $s_1|_{U} = s_2|_{U}$

Ex $\mathcal{O}_x$ = "germs of holomorphic functions at x" = convergent Taylor series
$\mathbb{Z}_x = \mathbb{Z}$

Given $f \in \mathcal{F}(U)$ and $x \in U$, let f_x be the image of f in $\mathcal{F}_x$.
$\varphi : \mathcal{F} \to \mathcal{G}$ induces a map on stalks, $\varphi : \mathcal{F}_x \to \mathcal{G}_x$.

Def Call $\varphi : \mathcal{F} \to \mathcal{G}$ surjective if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective $\forall x$.

Another way of expressing this: use sheafification — replace a presheaf by "smallest sheaf with the same stalks":

Def Given a presheaf $\mathcal{F}$, its sheafification $\mathcal{F}^+$ is the sheaf

$$\mathcal{F}^+(U) = \Big\{ \{s_x\} \in \prod_{x \in U} \mathcal{F}_x : \forall x \in U,\ \exists V \text{ with } U \supset V \ni x \text{ and } f \in \mathcal{F}(V) \text{ s.t. } \forall x' \in V,\ f_{x'} = s_{x'} \Big\}$$

[Fancy remark: sheafification is a functor

$PrSh(M) \longrightarrow Sh(M)$

which is left adjoint to the "forgetful" (inclusion) functor

$Sh(M) \longrightarrow PrSh(M)$. i.e. $Hom_{Sh}(\mathcal{F}^+, \mathcal{G}) = Hom_{PrSh}(\mathcal{F}, \mathcal{G})$

[cf. Stone-Čech compactification $Top \to KHaus$:
if K is compact $Hom(\overline{M}, K) = Hom(M, K)$]]

So now we can define sheaves $Im\,\varphi$, $Coker\,\varphi$ by sheafif^n:

$Im\,\varphi = (U \mapsto Im\,\varphi_U)^+$. (Naturally a subsheaf of $\mathcal{G}$.)

$Coker\,\varphi = (U \mapsto Coker\,\varphi_U)^+$

Prop Say $\varphi : \mathcal{F} \to \mathcal{G}$ hom. of sheaves.

φ surjective $\iff Im\,\varphi = \mathcal{G}$

$\iff Coker\,\varphi = 0$.

Pf Say φ is surjective. Then take any $g \in \mathcal{G}(U)$. Each $g_x \in Im\,\varphi_x$.

This means $g \in (Im\,\varphi)(U)$.

Other implications are similarly tautological. ∎

Ex $X = \mathbb{C}$, $\varphi : \mathcal{O} \to \mathcal{O}$, $f \mapsto zf$. $Coker\,\varphi$ = skyscraper sheaf S_0