Line bundles and divisors

Def $Y \subset X$ is a <u>hypersurface</u> if it is locally cut out by an equation $f_i = 0$, f_i holomorphic on U_i.

But not globally, since there might not even be nontriv. global hol f's on X!
Still, something analogous is true: we could take the local f_i's and stitch them together into a line bundle —

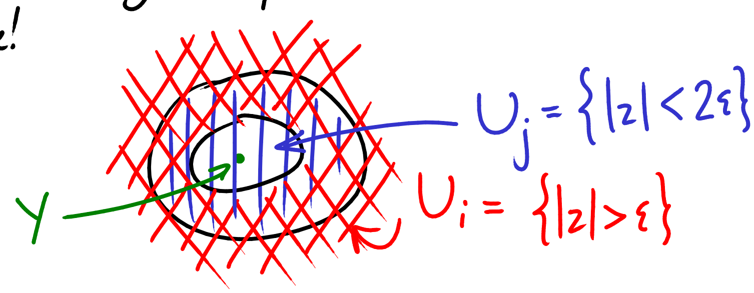
Def $\mathcal{O}(-Y)$ is the sheaf on X whose local sections are hol functions on X which vanish on Y.

Def/Prop $\mathcal{O}(-Y)$ is the sheaf of sections of a line bundle over X, which we also call $\mathcal{O}(-Y)$.

Pf It's a locally free sheaf of $\mathcal{O}$-modules over X.
(Generated by f_i on U_i). Explicitly: transition fncs. $\varphi_{ij} = f_i / f_j$. ∎

NB: $\mathcal{O}(-Y)$ is generally different from $\mathcal{O}$ even in the C^∞ sense!

e.g.



$\left[\varphi_{ij} = z/1 = z \text{ has nontriv. winding \# as a map } U_{ij} \to \mathbb{C}^\times. \right]$

Def $\mathcal{O}(Y) = \mathcal{O}(-Y)^*$.

Prop $\mathcal{O}(Y)$ admits a canonical global section s_Y whose zero locus is exactly Y.

Pf s_Y is the operation of evaluation, acting on sections of $\mathcal{O}(-Y)$.
It clearly vanishes on the fibers of $\mathcal{O}(-Y)$ over Y, doesn't vanish elsewhere. ∎

Def A <u>divisor</u> D is a formal Σ of hypersurfaces, with $\mathbb{Z}$-weights,

$$D = \sum_Y n_Y [Y] \qquad n_Y \in \mathbb{Z} \qquad (\text{locally finite})$$

Divisors form an abelian group in the obvious way: $n[Y] + m[Y] = (n+m)[Y]$.

Def $\mathcal{O}(D) = \bigotimes_Y \mathcal{O}(Y)^{\otimes n_Y}$.

Def Say $D \sim D'$, "D is linearly equivalent to D'", if $\mathcal{O}(D) \simeq \mathcal{O}(D')$.

Def If $f \in K(X)$ (meromorphic function)

- $\nu_Y(f)$ = order of vanishing of f along Y
 (defined locally by $f = f_Y^{\nu_Y(f)} g$ with $g \in \mathcal{O}^\times$ —
 <u>positive</u> if f has a zero, <u>negative</u> if f has a <u>pole</u> —
 <u>warning</u>: need some work to be sure this really defines
 s.t. well defined globally)

- $\operatorname{div}(f) = \sum_Y \nu_Y(f)\,[Y]$ (sum over <u>irreducible</u> hypersurfaces)

Prop $D \sim D' \iff \exists f \in K(X)$ with $\operatorname{div}(f) = D - D'$
$\iff \exists s \in \mathcal{O}(D)$ with $\operatorname{div}(s) = D'$.

Pf Exercise.

For X a curve, divisors are just <u>sets of points</u> weighed by integers, so

<u>Abel-Jacobi map</u> $X \longrightarrow \mathrm{Pic}(X)$

$z \longmapsto \mathcal{O}(z)$

(<u>Exercise</u>: Injective for X of genus >0, $\simeq$ for X of genus 1.)

(More generally have maps $X^d \to \mathrm{Pic}(X) \ldots$)

Prop $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.

Pf Easy.

Def Call $D = \sum n_Y [Y]$ <u>effective</u> if all $n_Y \geq 0$.

Ex On $\mathbb{CP}^1$, $D = \sum n_i [z_i]$ has $\mathcal{O}(D) \sim \mathcal{O}(\sum n_i)$.

e.g. $\mathcal{O}([z_1] - [z_2]) \simeq \mathcal{O}$, because the function $f(z) = \frac{z - z_1}{z - z_2}$ has $\mathrm{div}(f) = [z_1] - [z_2]$.

equivalently, $\mathcal{O}([z_1]) \simeq \mathcal{O}([z_2]) \simeq \mathcal{O}(1)$.

But on torus, $\mathcal{O}(p_1) \neq \mathcal{O}(p_2)$.

Normal bundles

$Y \subset X$ complex submfd: define $NY = T_{hol}X|_Y / T_{hol}Y$ (hol. bundles)

$$0 \to T_{hol}Y \to T_{hol}X|_Y \to NY \to 0$$

Prop If Y is a (smooth) hypersurface, then $NY \simeq \mathcal{O}(Y)|_Y$.

Pf Say Y cut out locally by $f_i = 0$

df_i is a local section of $(NY)^* \subset (T_{hol}X|_Y)^*$. Y smooth $\Rightarrow df_i \neq 0$.

Then
$$\begin{aligned} df_i &= d(\varphi_{ij} f_j) \\ &= d\varphi_{ij} f_j + \varphi_{ij} df_j \\ &= \varphi_{ij} df_j \quad \text{on } Y \end{aligned}$$

$\left[\varphi_{ij} = \frac{f_i}{f_j} \text{ nonvanishing regular}\right]$

Thus we have a well def. iso, $(NY)^* \simeq \mathcal{O}(-Y)|_Y$

$$df_i \mapsto f_i$$

$\blacksquare$