A good device for studying failure of local/global is sheaves where it doesn't occur:

Def $\mathcal{F}$ a sheaf: call $\mathcal{F}$ flabby if for any open $U \subset M$ the restriction map $r_{M,U}: \mathcal{F}(M) \to \mathcal{F}(U)$ is surjective.

Flabby sheaves are very convenient for defining cohomology;
But they don't occur so much in nature (at least in analytic topologies!)
(e.g. even C^0 isn't flabby... say $M = S^1$, $U = S^1 \setminus \{pt\}$)

From now on, M is Hausdorff paracompact.

Def For $S \subset M$ closed, define $\mathcal{F}(S) = \varinjlim_{U \supset S} \mathcal{F}(U)$.

Def $\mathcal{F}$ a sheaf: call $\mathcal{F}$ soft if for any closed subset $S \subset M$ the restriction map $r_{MS}: \mathcal{F}(M) \to \mathcal{F}(S)$ is surjective.

Ex Flabby sheaves are soft.

Soft sheaves do occur in nature:

Def Call $\mathcal{F}$ fine if for any loc. finite cover $\{U_i\}$ of M there are morphisms $\eta_i : \mathcal{F} \to \mathcal{F}$

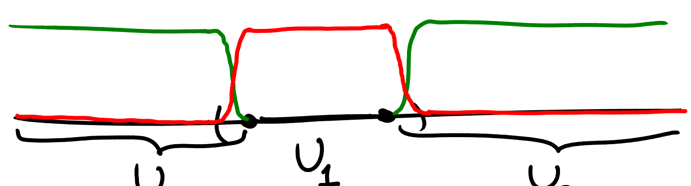
s.t. a) $\sum \eta_i = \mathbb{1}$.

b) $\eta_i(\mathcal{F}_x) = 0$ $\forall x$ in nbhd of $M \setminus U_i$.

Ex C^0, C^∞, $\Omega^{p,q}$, $C^\infty(E)$ are all fine sheaves (use partition of unity).

Prop $\mathcal{F}$ fine $\Rightarrow$ $\mathcal{F}$ soft.

Pf Given $f \in \mathcal{F}(S)$ cover S by U_i with $f_i \in \mathcal{F}(U_i)$ s.t. $f_i|_{S \cap U_i} = f|_{S \cap U_i}$
and also take $U_0 = M \setminus S$. Each $\eta_i(f_i)$ extends to the whole M.
Then $\sum_i \eta_i(f_i)$ is the desired extension of f to M. ∎

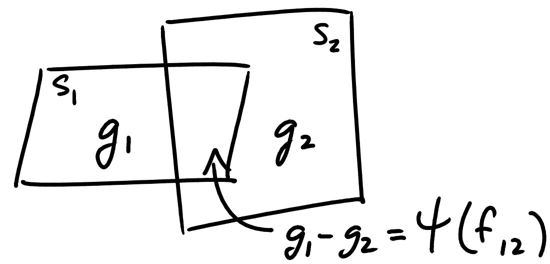


Lemma If $0 \to \mathcal{F} \xrightarrow{\psi} \mathcal{G} \xrightarrow{\varphi} \mathcal{H} \to 0$ exact and $\mathcal{F}$ is soft, then

$$0 \to \mathcal{F}(M) \to \mathcal{G}(M) \to \mathcal{H}(M) \to 0$$

is also exact.

Pf idea Just need to check surjectivity at $\mathcal{H}(M)$. Take $h \in \mathcal{H}(M)$.
Using Hausdorff pcpt, $\exists$ loc. fin. cover of M by closed S_i for which $h|_{S_i} = \varphi(g_i)$
Then pb is to glue the g_i together.



On $S_1 \cap S_2$ we have $g_1 - g_2 = \psi(f_{12})$
So take $\tilde{g}_2 = g_2 + \psi(\hat{f}_{12})$ where $\hat{f}_{12}$ is extension of f_{12} to S_2
Then $g_1, \tilde{g}_2$ do agree — can be glued. ∎

Lemma $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact, $\mathcal{F}, \mathcal{G}$ soft $\Rightarrow \mathcal{H}$ soft

Pf $0 \to \mathcal{F}(S) \to \mathcal{G}(S) \to \mathcal{H}(S) \to 0$
$h \in \mathcal{H}(S)$ comes from $g \in \mathcal{G}(S)$, which extends to $g \in \mathcal{G}(M)$, then its image is the desired ext^n of h. ∎

Prop If $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots$, exact, all $\mathcal{F}_i$ soft

Then $0 \to \mathcal{F}_0(M) \to \mathcal{F}_1(M) \to \mathcal{F}_2(M) \to \cdots$ is exact.

Pf "Splice in"

$\mathcal{K}_1 = \mathcal{F}_0 \hookrightarrow \mathcal{F}_1$

$\mathcal{K}_2 = \mathcal{F}_1 / \mathcal{K}_1 \hookrightarrow \mathcal{F}_2$

$\mathcal{K}_3 = \mathcal{F}_2 / \mathcal{K}_2 \hookrightarrow \mathcal{F}_3$

$\vdots$

$$\begin{tikzcd}
& & & 0 & & & & 0 \\
& & \mathcal{K}_1 \arrow[ur] \arrow[dr] & & & & \mathcal{K}_3 \arrow[ur] \arrow[dr] & \\
0 \arrow[r] & \mathcal{F}_0 \arrow[ur] \arrow[rr] & & \mathcal{F}_1 \arrow[rr] \arrow[dr] & & \mathcal{F}_2 \arrow[rr] \arrow[ur] & & \cdots \\
& & & & \mathcal{K}_2 \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Exactness of these short seq. is equivalent to that of the original long seq.

All $\mathcal{K}_i$ are soft, by the lemma.

So, apply to M, still have short exact seq.

Thus get exactness for the original long exact seq.