# Sheaf cohomology

## Godement resolution

$\mathcal{F}$ a sheaf (of ab gps) over $M$:

Define $C^0(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x$. $C^0(\mathcal{F})$ is flabby, $0 \to \mathcal{F} \to C^0(\mathcal{F})$.

Now inductively set

$$D^0(\mathcal{F}) = \mathcal{F}, \quad D^i(\mathcal{F}) = C^{i-1}(\mathcal{F}) / D^{i-1}(\mathcal{F}), \quad C^i(\mathcal{F}) = C^0(D^i(\mathcal{F}))$$

so $\quad 0 \to D^i(\mathcal{F}) \to C^i(\mathcal{F}) \to D^{i+1}(\mathcal{F}) \to 0$

and we splice them together into $\quad 0 \to \mathcal{F} \to C^0(\mathcal{F}) \to C^1(\mathcal{F}) \to C^2(\mathcal{F}) \to \cdots$

ie $\quad 0 \to \mathcal{F} \xrightarrow{\varphi_0} C^0(\mathcal{F}) \xrightarrow{\varphi_1} C^0(\operatorname{coker} \varphi_0) \to C^0(\operatorname{coker} \varphi_1) \to \cdots$

A canonical flabby resolution.

**Def** Sheaf cohomology $H^\bullet(M, \mathcal{F})$ is the homology of the cochain complex

$$0 \to C^0(\mathcal{F})(M) \to C^1(\mathcal{F})(M) \to \cdots$$

**Prop**

a) 1) $H^0(M,\mathcal{F}) = \mathcal{F}(M)$.

2) If $\mathcal{F}$ is soft, then $H^q(M,\mathcal{F}) = 0$ for $q>0$.

b) Sheaf morphism $\varphi: \mathcal{F} \to \mathcal{G}$ induces $\varphi_q: H^q(M,\mathcal{F}) \to H^q(M,\mathcal{G})$ such that

1) $\varphi_0 = \varphi_M$

2) If $\varphi = \mathbb{1}$ then $\varphi_q = \mathbb{1}$ $(q \geq 0)$

3) $\varphi_q \circ \psi_q = (\varphi \circ \psi)_q$

c) Given $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$

there is $\delta^q: H^q(M,\mathcal{H}) \to H^{q+1}(M,\mathcal{F})$ s.t.

1)
$$0 \to H^0(M,\mathcal{F}) \to H^0(M,\mathcal{G}) \to H^0(M,\mathcal{H}) \xrightarrow{\delta^0} \cdots$$
$$\to H^1(M,\mathcal{F}) \to H^1(M,\mathcal{G}) \to H^1(M,\mathcal{H}) \xrightarrow{\delta^1} \cdots$$
$$\longrightarrow \cdots$$
is exact

2)
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F} & \to & \mathcal{G} & \to & \mathcal{H} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{F}' & \to & \mathcal{G}' & \to & \mathcal{H}' & \to & 0 \end{array}$$
induces a comm. dig. of long exact sequences in the obvious way

**Pf**

a) 1) Since $0 \to \mathcal{F} \to C^0(\mathcal{F}) \to C^1(\mathcal{F})$ is exact, so is $0 \to \mathcal{F}(M) \to C^0(\mathcal{F})(M) \to C^1(\mathcal{F})(M)$ so $\mathcal{F}(M)$ is kernel of $C^0(\mathcal{F})(M) \to C^1(\mathcal{F})(M)$ as desired.

2) use exactness of $0 \to \mathcal{F}(M) \to C^0(\mathcal{F})(M) \to C^1(\mathcal{F})(M) \to \cdots$ since all sheaves here are soft

b) construct a map of cochain complexes from $\varphi$ easily, which then gives the desired map on cohomology

c) given
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$
the induced maps of cochain complexes
$$0 \to C^\cdot(\mathcal{F})(M) \to C^\cdot(\mathcal{G})(M) \to C^\cdot(\mathcal{H})(M) \to 0$$
are also exact, since all the sheaves $C^\cdot(\cdot)$ are soft.

Then it's a general homological-algebra construction:

$$\beta \qquad\qquad \alpha = \varphi(\beta),\ d\alpha = 0$$

$$\begin{tikzcd}
0 \arrow[r] & C^p(\mathcal{F})(M) \arrow[r] \arrow[d] & C^p(\mathcal{G})(M) \arrow[r, "\varphi"] \arrow[d] & C^p(\mathcal{H})(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{p+1}(\mathcal{F})(M) \arrow[r, "\psi"] & C^{p+1}(\mathcal{G})(M) \arrow[r, "\varphi"] & C^{p+1}(\mathcal{H})(M) \arrow[r] & 0
\end{tikzcd}$$

$$\gamma \qquad\qquad d\beta$$

$\psi(d\gamma) = d(\psi(\gamma)) = 0$ $\Rightarrow d\gamma = 0$ $\qquad$ $\varphi(d\beta) = 0 \Rightarrow d\beta = \psi(\gamma)$

we define $\delta^p(\alpha)$ to be $[\gamma]$. (Have to check it's indep. of choices.) $\blacksquare$

**Def** Call resolution $0 \to \mathcal{F} \to \mathcal{A}^\bullet$ acyclic if $H^p(M, \mathcal{A}^q) = 0$ $\forall p > 0,\ q \geq 0$.

We showed soft resolutions are acyclic.

**Prop** (Abstract de Rham Thm)

Given a resolution $0 \to \mathcal{F} \to \mathcal{A}^\bullet$ there is a natural hom. $\sigma^p : H^p(\mathcal{A}^\bullet(X)) \to H^p(X, \mathcal{F})$

If the reso. is acyclic then $\sigma^p$ is $\simeq$.

This is the main tool one uses in practice to compute cohomology!

**Cor** (deRham) $H^p(M,\mathbb{R}) \simeq H^p(\Omega^\bullet(M)) = \dfrac{\ker\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right)}{\operatorname{im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right)}$

$\simeq H^p(S^\bullet(M,\mathbb{R}))$ [singular cochains]

**Cor** (Dolbeault) $H^q(X,\Omega^p_{hol}) \simeq H^q(\Omega^{p,\bullet}(X)) = \dfrac{\ker\left(\bar\partial : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)\right)}{\operatorname{im}\left(\bar\partial : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X)\right)}$

**Pf** We'll give a chain of maps $\dfrac{\ker d \text{ on } A^p(M)}{\operatorname{im} d \text{ on } A^{p-1}(M)} \xrightarrow{\sim} H^1(\cdot) \xrightarrow{\sim} H^2(\cdot) \to \cdots \to H^p(M,F)$.

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, equal] & A^0 \arrow[rr] \arrow[dr] & & A^1 \arrow[r] & \cdots \arrow[r] & A^{p-1} \arrow[rr] \arrow[dr] & & A^p \arrow[rr] \arrow[dr] & & A^{p+1} \arrow[r] & \cdots \\
& K^0 \arrow[ur] & & K^1 \arrow[ur] & & & & K^p \arrow[ur] & & K^{p+1} \arrow[ur] & &
\end{tikzcd}$$

$K^p = \operatorname{Ker}(A^p \to A^{p+1})$

$$0 \to K^p \to A^p \to K^{p+1} \to 0$$

replace $p \to p-r$:

$$0 \to K^{p-r} \to A^{p-r} \to K^{p+1-r} \to 0$$

Long ex seq $\rightsquigarrow$ $\gamma^p_r : H^{r-1}(M, K^{p-r+1}) \to H^r(M, K^{p-r})$

It's $\simeq$ if $A^\bullet$ acyclic, except at $r=1$. At $r=1$ we get instead

$$H^0(M, A^{p-1}) \to H^0(M, K^p) \xrightarrow{\gamma^p_1} H^1(M, K^{p-1}) \to 0$$

ie, $\gamma^p_1 : \dfrac{H^0(M, K^p)}{\operatorname{Im}(H^0(A^{p-1}) \to H^0(K^p))} \xrightarrow{\sim} H^1(M, K^{p-1})$

So, have to think a little about what this map actually is.

$H^0(M, K^p) = \ker\left(H^0(M, A^p) \to H^0(M, A^{p+1})\right)$ and the map is the obvious one taking $H^0(A^{p-1})$ to this kernel — i.e. $\tilde\gamma^p_1 : \dfrac{\operatorname{Ker}(H^0(A^p) \to H^0(A^{p+1}))}{\operatorname{Im}(H^0(A^{p-1}) \to H^0(A^p))} \xrightarrow{\sim} H^1(K^{p-1})$

$\qquad \parallel$

$\qquad H^p(A^\bullet(M))$

Then define $\sigma^p = \gamma^p_p \circ \gamma^p_{p-1} \circ \cdots \circ \gamma^p_2 \circ \tilde\gamma^p_1 :$

$$H^p(A^\bullet(M)) \xrightarrow{\sim} H^1(K^{p-1}) \xrightarrow{\sim} H^2(K^{p-2}) \to \cdots \to H^p(K^0) = H^p(M, F)$$

Philosophy: Suppose you want to compute $H^{\bullet}(\mathcal{F})$.
How to do it?

Embed $\mathcal{F}$ in some acyclic sheaf:

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{A}/\mathcal{F} \to 0$$

gives

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{A}) \to H^0(\mathcal{A}/\mathcal{F}) \to H^1(\mathcal{F}) \to \cancel{H^1(\mathcal{A})}^{\,0}$$
$$\to H^1(\mathcal{A}/\mathcal{F}) \to H^2(\mathcal{F}) \to \cancel{H^2(\mathcal{A})}^{\,0}$$
$$\vdots$$

That's enough to determine $H^{\bullet}(\mathcal{F})$ inductively!