Curvature

Recall a connection $\nabla: \Omega^0(E) \to \Omega^1(E)$ naturally extends to

$$\nabla: \Omega^k(E) \to \Omega^{k+1}(E) \qquad \text{(sometimes called } d_\nabla)$$

by $\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \nabla s$

It obeys $\nabla(\beta \wedge t) = d\beta \wedge t + (-1)^{|\alpha|} \beta \wedge \nabla(t)$

Def/Prop F_∇ is the unique elt in $\Omega^2(\text{End } E)$ such that $(\nabla \circ \nabla)s = F_\nabla \cdot s$ for $s \in \Omega^0(E)$.

Ex 1) If E is trivial, $\nabla = d + A$, then $F_\nabla = dA + A \wedge A$ (Exercise)

[Here "$\wedge$" on $A^\cdot(\text{End } \mathbb{C}^r)$ is defined by $(\alpha \otimes A) \wedge (\alpha' \otimes A') = (\alpha \cdot \alpha') \otimes (A \wedge A')$]

2) If $a \in \Omega^1(\text{End } E)$ then $F_{\nabla + a} = F_\nabla + \nabla a + a \wedge a$

(this generalizes #1)

Prop $(\nabla \circ \nabla)\omega = F_\nabla \wedge \omega$ for any $\omega \in A^\cdot(E)$.

Pf Exercise. (Just use def of ∇.)

Conn. $\nabla_{1,2}$ on $E_{1,2}$ induce

- ∇ on $E_1 \oplus E_2$ w/ $F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}$
- ∇ on $E_1 \otimes E_2$ w/ $F_\nabla = F_{\nabla_1} \otimes F_{\nabla_2}$
- ∇ on E_1^* w/ $F_\nabla = -F_{\nabla_1}^T$

In p^r, get a conn. ∇ on $E \otimes E^* = \text{End}(E)$. It obeys

$$\nabla(\xi s) = \nabla(\xi) \cdot s + \xi \cdot \nabla(s) \qquad \xi \in A^0(\text{End } E),\ s \in A^0(E)$$

Lemma (Bianchi): $\nabla(F_\nabla)=0$.

Pf $\nabla(F_\nabla)(s)=\nabla(F_\nabla\wedge s)-F_\nabla\wedge(\nabla(s))=\nabla(\nabla\circ\nabla(s))-\nabla\circ\nabla(\nabla(s))=0$.

[So Bianchi is a kind of associativity...]

Prop 1) If ∇ is a Hermitian conn. on (E,h) then $F_\nabla\in\Omega^2(X,\mathrm{End}(E,h))$. (real v.s.)

2) If ∇ is compatible with hol str on $(E,\bar\partial_E)$ then $F_\nabla\in\Omega^{2,0}\oplus\Omega^{1,1}(X,\mathrm{End}(E))$

3) If ∇ is Chern connection on $(E,h,\bar\partial_E)$ then $F_\nabla\in\Omega^{1,1}_{\mathbb{R}}(X,\mathrm{End}(E,h))$.

↑ defined as $\Omega^{1,1}(X,\mathrm{End}(E))\cap\Omega^2(X,\mathrm{End}(E,h))$

Pf 1) Local unitary triv: $\nabla=d+A$, $A^*=-A$, $A\in\Omega^1(\mathrm{End}\,\mathbb{C}^n)$

$$F_\nabla=dA+A\wedge A$$
$$F_\nabla^*=dA^*+(A\wedge A)^*$$
$$=dA^*-A^*\wedge A^*$$
$$=-dA-A\wedge A$$
$$=-F_\nabla$$

[for A,A' decomposable
$A\wedge A'=(\alpha\otimes B)\wedge(\alpha'\otimes B')$
$=(\alpha\wedge\alpha')\otimes BB'$
$(A\wedge A')^*=(\alpha\wedge\alpha')\otimes B'^*B^*$
$=-(\alpha'\wedge\alpha)\otimes B'^*B^*$
$=-(\alpha'\otimes B'^*)\wedge(\alpha\otimes B^*)$
$=-(A'^*\wedge A^*)$]

2) Local hol triv: $\nabla=d+A$ with $A\in\Omega^{1,0}(\mathrm{End}\,\mathbb{C}^n)$

$$F_\nabla=dA+A\wedge A=\underbrace{\bar\partial A}_{\Omega^{1,1}}+\underbrace{(\partial A+A\wedge A)}_{\Omega^{2,0}}$$

3) Combine 1,2. $F_\nabla\in\Omega^{2,0}\oplus\Omega^{1,1}(\mathrm{End}\,E)\Rightarrow F_\nabla^*\in\Omega^{0,2}\oplus\Omega^{1,1}(\mathrm{End}\,E)$

then using $F_\nabla=-F_\nabla^*$ gives $F_\nabla\in\Omega^{1,1}(\mathrm{End}\,E)$

and moreover $F_\nabla\in\Omega^2(\mathrm{End}(E,h))$

combining these gives $F_\nabla\in\Omega^{1,1}_{\mathbb{R}}(\mathrm{End}(E,h))$ ∎

Ex In local hol. triv, Chern connection $\nabla=d+A$, $F_\nabla=dA+A\wedge A=\bar\partial A$.

Indeed $A=\bar H^{-1}\partial\bar H$, so $F_\nabla=\bar\partial(\bar H^{-1}\partial\bar H)$.

Now suppose ∇ is Chern connection in $(E, \bar\partial, h)$.

Prop $[F_\nabla] \in H^{1,1}(X, \mathrm{End}\,E)$ is independent of h, depends only on $(E, \bar\partial)$.

Pf By Bianchi, $0 = \nabla(F_\nabla)$. In pt, since $\nabla^{(0,1)} = \bar\partial$, F_∇ is $\bar\partial$-closed.

So $[F_\nabla] \in H^{1,1}(X, \mathrm{End}\,E)$.

A change of h induces

$$\nabla \to \nabla + \eta \text{ for } \eta \in \Omega^{1,0}(X), \quad F_\nabla \to F_\nabla + d\eta + \eta\wedge\eta = F_\nabla + \bar\partial\eta \qquad \blacksquare$$

This is also a consequence of:

Prop $[F_\nabla] = A(E)$.

Pf Take local triv. $\varphi_i : E \to \mathbb{C}^r$ over U_i.

2 different resolutions of $\Omega^1 \otimes \mathrm{End}\,E$ fit naturally into a double complex, w/ acyclic columns:

$$\begin{array}{ccccccc}
\Omega^1\otimes \mathrm{End}\,E & \longrightarrow & C^0(\Omega^1\otimes \mathrm{End}\,E) & \longrightarrow & \boxed{C^1(\Omega^1\otimes \mathrm{End}\,E)} \ \leftarrow A(E) & \longrightarrow & \cdots \\
\downarrow i & & \downarrow i & & \downarrow i & & \\
\mathcal{A}^{1,0}(\mathrm{End}\,E) & \longrightarrow & C^0(\mathcal{A}^{1,0}(\mathrm{End}\,E)) \ \textcircled{2} & \xrightarrow{\delta_1} & C^1(\mathcal{A}^{1,0}(\mathrm{End}\,E)) \ \textcircled{3} & \longrightarrow & \cdots \\
\downarrow \bar\partial & & \downarrow \bar\partial & & & & \\
\boxed{\mathcal{A}^{1,1}(\mathrm{End}\,E)} \ \leftarrow F_\nabla & \xrightarrow{\delta_0} & C^0(\mathcal{A}^{1,1}(\mathrm{End}\,E)) \ \textcircled{1} & \longrightarrow & \cdots & & \\
\downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & & &
\end{array}$$

To compare the two classes, use "stair-step" construction

$$F_\nabla = \varphi_i^{-1}\left(\bar\partial(\bar H_i^{-1}\partial \bar H_i)\right)\varphi_i \quad \forall i$$

so $\delta_0(F_\nabla)$ is the 0-cochain $\varphi_i^{-1}(\bar\partial(\bar H_i^{-1}\partial\bar H_i))\varphi_i$ on U_i ①

this is $\bar\partial$ of $\varphi_i^{-1}(\bar H_i^{-1}\partial\bar H_i)\varphi_i$ ②

applying δ_1 gives 1-cochain $\varphi_i^{-1}(\bar H_i^{-1}\partial\bar H_i)\varphi_i - \varphi_j^{-1}(\bar H_j^{-1}\partial \bar H_j)\varphi_j$ on U_{ij} ③

$$= \varphi_j^{-1}\left(\bar H_j^{-1}\partial\bar H_j - \varphi_{ij}^{-1}(\bar H_i^{-1}\partial\bar H_i)\varphi_{ij}\right)$$

$$= \Psi_j^{-1}(\Psi_{ij}^{-1}\partial\Psi_{ij})\Psi_j \qquad \left[\text{using } \Psi_{ij}^t H_i \bar{\Psi}_{ij} = H_j\right]$$

which is indeed what we get by applying i to $A(E)$. ∎

Positivity

Def $\alpha \in \Omega^{1,1}_{\mathbb{R}}$ is (semi)positive if $\forall v \in T^{1,0}X$, $v \neq 0$,

$$-i\alpha(v,\bar{v}) > 0 \quad (\geq 0)$$

Ex Any Kähler form ω is positive.

Q: Given a Hermitian hol. line bundle, when is F_∇ positive?

Def $\mathcal{L}$ hol l.b. over X: $\mathcal{L}$ is globally generated if $\forall x \in X$, $\exists s \in H^0(X,\mathcal{L})$ with $s(x) \neq 0$.

If $\mathcal{L}$ is globally generated hol l.b. then choosing a basis $\{s_i\}$ for $H^0(\mathcal{L})$, we can define a Hermitian metric on $\mathcal{L}$ by

$$h(s) = \frac{|s|^2}{\sum_i |s_i|^2} \quad \text{(in any hol. loc. triv.)} \qquad (*)$$

Ex On $X = \mathbb{CP}^n$, $\mathcal{L} = \mathcal{O}(1) = \mathcal{O}(-1)^*$ is glob. gen. by $n+1$ sections $z_0, \ldots, z_n$.
Corresp. h is rep^d in patch U_i by $H = \dfrac{|z_i|^2}{\sum_{0\leq k\leq n} |z_k|^2} = \dfrac{1}{1+\sum_{0\leq k\leq n,\, i\neq k} |w_k|^2}$,
and $i\bar{\partial}\partial \log H = \omega_{FS}$ (Kähler form).

Prop If $\mathcal{L}$ is globally generated, the Chern connection assoc to $(*)$ is semipositive.

Pf Let $V = H^0(\mathcal{L})$. There's canonical map $\Psi: \mathcal{L}^* \to V^*$
$$\ell \mapsto (\text{evaluation at } \ell)$$

It's linear $\simeq$ on each fiber of $\mathcal{L}$ $\Rightarrow$ can view it as a map of line bundles

$$\begin{array}{ccc} \mathcal{L}^* & \xrightarrow{\psi} & V^* = \mathcal{O}(-1) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & \mathbb{P}(V^*) \end{array}$$

this identifies $\mathcal{L}^* \simeq \varphi^* \mathcal{O}(-1)$ i.e. $\mathcal{L} \simeq \varphi^* \mathcal{O}(1)$.

Now take a basis $\{s_1, \dots, s_n\}$ in $V = H^0(\mathcal{L}) \simeq H^0(\mathcal{O}(1))$.

This induces metrics in both bundles, related by pullback.

So $F_\nabla = \varphi^* \omega_{FS}$,

ie $F_\nabla(v, \bar{v}) = \omega_{FS}(\varphi_* v, \varphi_* \bar{v})$, $\Rightarrow$ semipositivity for F_∇ ∎

Similar notion for <u>vector</u> bundles:

<u>Def</u> (E,h) Hermitian, ∇ Herm, $F_\nabla \in \mathcal{A}^{1,1}(X, \operatorname{End} E)$:

F_∇ is <u>positive</u> (semipositive) if $h(F_\nabla(s), s)(v, \bar{v}) > 0$ $(\geq)$ $\forall s \in \mathcal{A}^0(E)$, $v \in T^{1,0}X$, $v \neq 0$

<u>Prop</u> If E is globally generated hol vb, then a basis for $H^0(E)$ yields a Herm metric in E; let ∇ be Chern conn; then F_∇ is semipositive.

<u>Pf</u> See Huybrechts.