Interpretations of c_1

We've had 2 <u>different</u> definitions of c_1 so far.

Given a hol. line bundle $\mathcal{L} \to X$ we previously defined $c_1(\mathcal{L})$ using the <u>exponential sequence</u>

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^{\times} \to 0$$

inducing

$$\begin{array}{c} H^1(X, \mathcal{O}^{\times}) \to H^2(X, \underline{\mathbb{Z}}) \\ \shortparallel \\ \mathrm{Pic}(X) \end{array}$$

For C^∞ bundles, can do similarly: let $\mathcal{C}$ be sheaf of C^∞ f'n's
$\mathcal{C}^{\times}$ " " " <u>invertible</u> C^∞ f'n's

Then $H^1(M, \mathcal{C}^{\times})$ parameterizes $\simeq$ classes of C^∞ line bundles.

Exp. seq.

$$0 \to \underline{\mathbb{Z}} \to \mathcal{C} \to \mathcal{C}^{\times} \to 0$$

induces

$$\cdots \to \overset{0}{\cancel{H^1(M, \mathcal{C})}} \to H^1(M, \mathcal{C}^{\times}) \xrightarrow{\delta} H^2(M, \underline{\mathbb{Z}}) \to \overset{0}{\cancel{H^2(M, \mathcal{C})}} \to \cdots$$

i.e. $\quad H^1(M, \mathcal{C}^{\times}) \simeq H^2(M, \underline{\mathbb{Z}})$.

<u>Prop</u> L C^∞ l.b. over M: $\quad \delta(L) = -c_1(L)$.

<u>Pf</u>

$$\begin{tikzcd}
 & \mathbb{C} \arrow[r]\arrow[d] & \check{C}^0(\mathbb{C}) \arrow[r] & \check{C}^1(\mathbb{C}) \arrow[r] & \check{C}^2(\mathbb{C}) \arrow[d] & \delta(L) \arrow[l, squiggly] \\
 & \Omega^0 \arrow[d] & & (4)\ \check{C}^1(\Omega^0) \arrow[r]\arrow[d] & (5)\ \check{C}^2(\Omega^0) & \\
 & \Omega^1 \arrow[d] & (2)\ \check{C}^0(\Omega^1) \arrow[r]\arrow[d, "d"] & (3)\ \check{C}^1(\Omega^1) & & \\
c_1(L) \arrow[r, squiggly] & \Omega^2 \arrow[r] & (1)\ \check{C}^0(\Omega^2) & & &
\end{tikzcd}$$

$[\check{C} = \check{\text{C}}\text{ech cocycles}]$

Choose a conn. ∇ in L, and trivializ^n over U_i w/ transition func $\psi_{ij} = e^{2\pi i \varphi_{ij}}$.

Then:

$$
\begin{array}{lll}
c_1(L) \longrightarrow & \frac{i}{2\pi} F_\nabla \text{ on } U_i & \text{(1)} \\
& \quad \| & \\
& d\left(\frac{i}{2\pi} A_i\right) \text{ on } U_i & \text{(2)} \\
& \quad \downarrow & \\
& \frac{i}{2\pi}(A_i - A_j) \text{ on } U_{ij} & \text{(3)} \\
& \quad \| & \\
& d(-\varphi_{ij}) \text{ on } U_{ij} & \text{(4)} \\
& \quad \downarrow & \\
-\delta(L) \longrightarrow & \varphi_{jk} - \varphi_{ik} + \varphi_{ij} \text{ on } U_{ijk} & \text{(5)}
\end{array}
$$

The standard diagram chase defining the map δ (exercise) shows this is indeed $-\delta(L)$. ∎

What this means: our def. of c_1 as $\left[\frac{i}{2\pi} F_\nabla\right]$ agrees with earlier Čech def.

In p$^{\text{the}}$, $c_1(L) \in H^2(M, \mathbb{Z})$ (or more exactly $\mathrm{Im}[H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})]$)

[Using splitting principle one could also show all $c_k(L) \in H^{2k}(M, \mathbb{Z})$.]

Now, yet another interpretation: suppose X compact, Y irred smooth hypersurface.

Consider the map

$$
\begin{aligned}
\int_Y : H^{2n-2}(X, \mathbb{R}) &\longrightarrow \mathbb{R} \\
\alpha &\longmapsto \int_Y \alpha
\end{aligned}
$$

Def/Prop By nondegeneracy of the pairing $H^2(X,\mathbb{R}) \times H^{2n-2}(X,\mathbb{R}) \to \mathbb{R}$ (Poincaré duality)

there exists $[Y] \in H^2(X, \mathbb{R})$ such that $\int_Y \alpha = \int_X \alpha \wedge [Y]$. ("fundamental class")

Prop $c_1(\mathcal{O}(Y)) = [Y]$.

Pf Let $s \in H^0(\mathcal{O}(Y))$ have $\mathrm{div}(s) = Y$. Choose a Herm. metric on $\mathcal{O}(Y)$, ∇ Chern conn; so away from Y, $F_\nabla = \partial\bar\partial \log \|s\|^2$ ($\|s\|^2 = h(s,s)$). Then, for α closed,

$$\frac{i}{2\pi}\int_X F_\nabla \wedge \alpha = \lim_{\varepsilon \to 0} \frac{i}{2\pi}\int_{X \setminus D_\varepsilon} \partial\bar\partial \log \|s\|^2 \wedge \alpha \qquad D_\varepsilon \text{ tubular nbhd of } Y$$

$$= \lim_{\varepsilon \to 0} \frac{i}{4\pi}\int_{X \setminus D_\varepsilon} d(\partial - \bar\partial)\log\|s\|^2 \wedge \alpha$$

$$= \lim_{\varepsilon \to 0} \frac{i}{4\pi}\int_{\partial D_\varepsilon} (\partial - \bar\partial)\log\|s\|^2 \wedge \alpha .$$

To get oriented, consider 1-d case, $Y = \{0\}$: $\|s\| = h(z,\bar z)\cdot |z|^2$ $h > 0$

Then $\displaystyle\lim_{\varepsilon \to 0} \frac{i}{4\pi}\int_{\partial D_\varepsilon} (\partial - \bar\partial)\left[\log h + \log z + \log \bar z\right] = \frac{i}{4\pi}\int_{\partial D_\varepsilon}\left[\frac{dz}{z} - \frac{d\bar z}{\bar z}\right] + \underbrace{\frac{i}{4\pi}\int_{\partial D_\varepsilon}(\partial - \bar\partial)\log h}_{\to 0 \text{ as } \varepsilon \to 0}$

$$= 1.$$

(minus sign b/c ∂D_ε is oriented "backwards")

The general case is similar, just more notation: perform the integral over S^1 by residue thm, to reduce to integral over Y,

$$= \int_Y \alpha .$$

(See Huyb. for details) ∎