Hirzebruch-Riemann-Roch formula

One more way of packaging characteristic classes:

Def Define $\tilde{P}_k$ by $\dfrac{\det(tB)}{\det(1-e^{-tB})} = \sum t^k \tilde{P}_k(B)$

Todd classes:

$$td_k(E,\nabla) = \tilde{P}_k\left(\tfrac{i}{2\pi}F_\nabla\right)$$

$$td_k(E) = [td_k(E,\nabla)] \in H^{2k}(M,\mathbb{C})$$

$$td(E) = \sum_{k=0}^{\infty} td_k(E)$$

$$= 1 + \tfrac{1}{2}c_1(E) + \tfrac{1}{12}\left(c_1(E)^2 + c_2(E)\right) + \cdots$$

$$td(X) = td(TX)$$

Def E a hol v.b. over compact X, $\dim_{\mathbb{C}} X = m$:

$$h^i(E) = \dim H^i(X,E)$$

$$\chi(E) = \sum_{i=0}^{m} (-1)^i h^i(E).$$

Thm $\chi(E) = \int_X ch(E)\, td(X).$ [Hirzebruch-Riemann-Roch]

(This really means $\int_X [ch(E)\, td(X)]_{2m}$.)

Rk • If $m=1$, this becomes

$$\chi(E) = \int_X (\operatorname{rank}(E) + c_1(E)) \cdot \left(1 + \tfrac{1}{2}c_1(X)\right) = \int_X c_1(E) + \tfrac{1}{2}\operatorname{rank}(E)\cdot c_1(X).$$

The first term $\int_X c_1(E)$ is conventionally called $\deg(E)$.

The second term $\int_X c_1(X) = 2-2g$ where g is the genus of X.

[Why? Look at the special case $E =$ trivial line bundle, $\mathcal{O}$.
Then $\chi(\mathcal{O}) = \frac{1}{2}\int_X c_1(X)$. But $\chi(\mathcal{O}) = h^0(\mathcal{O}) - h^1(\mathcal{O})$
$= 1 - g$.

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$
]

So, we get $h^0(E) - h^1(E) = \deg(E) + \operatorname{rank}(E)\cdot(1-g)$ [Riemann-Roch]

- Given $0 \to E \to F \to G \to 0$ one has $\chi(F) = \chi(E) + \chi(G)$ using long exact seq in cohomology. HRR formula relates this to the fact $ch(F) = ch(E) + ch(G)$.

- HRR formula shows that $\chi(E)$ depends only on C^∞ structure of E, not on its holomorphic structure. The individual $h^i(E)$ <u>can</u> depend on hol structure, though. (e.g. in case of degree 0 bundles on torus, generally have $h^0 = h^1 = 0$, but for trivial bundle have $h^0 = h^1 = 1$)

<u>Def</u> X compact complex: <u>χ_y-genus</u> of X is

$$\chi_y = \sum_{p=0}^{m} \chi(\Omega^p)\, y^p = \sum_{p=0}^{m} (-1)^q h^{p,q}(X)\, y^p$$

- $\chi_y(y=-1) = \sum (-1)^{p+q} h^{p,q}(X) = \chi(X)$.
- $\chi_y(y=0) = \chi(\mathcal{O})$
- $\chi_y(y=1) = \operatorname{sgn} X$ for X Kähler, m even

To compute χ_y, use

Meta-Prop Any identity among characteristic classes that holds when $E = \bigoplus L_i$ holds in general.

Pf Diagonalize F.

Thus: say $E = \bigoplus L_i$, $\gamma_i = c_1(L_i)$, then e.g. $c(E) = \prod (1+\gamma_i)$
("Chern roots")

$$ch(E) = \sum e^{\gamma_i}$$

$$td(E) = \prod \frac{\gamma_i}{1-e^{-\gamma_i}}$$

Prop $\displaystyle \chi_y = \int_X \prod \left(1+ye^{-\gamma_i}\right) \frac{\gamma_i}{1-e^{-\gamma_i}}$ where γ_i are Chern roots for TX.

Pf Just need $\displaystyle \sum_p ch(\Omega^p)\, y^p = \prod \left(1+ye^{-\gamma_i}\right)$

[Idea: $T^* = \bigoplus_i L_i^*$, then $\bigoplus \Omega^p = \bigotimes_i (\mathcal{O} \oplus L_i)$
and formally $\bigoplus \Omega^p y^p = \bigotimes_i (\mathcal{O} \oplus y L_i)$] ∎

Cor $\displaystyle \chi(X) = \int_X c_m(X)$.

Pf Take $y=-1$ in above: $\displaystyle \chi(X) = \chi_y(y=-1) = \int_X \prod \gamma_i = \int_X c_m(X)$.

So in particular, K3 surface has Euler characteristic $= \underline{24}$.