Calabi-Yau Manifolds

Kähler is an example of *special holonomy*: $\mathrm{Hol}_x(\nabla) \subset U(T_xX) \simeq U(n)$.
What if the holonomy is further reduced to $SU(T_xX) \simeq SU(n)$?

$\mathrm{Hol}_x(\nabla) \subset SU(T_xX)$ means it acts trivially on $\det(T_x^{1,0*}X) = K$.

Thus any nonzero element of K_x can be extended to a covariantly constant, nowhere vanishing section Ω.

$$\nabla\Omega = 0 \Rightarrow \bar{\partial}\Omega = 0$$

Thus

Prop If $\mathrm{Hol}_x(\nabla) \subset SU(T_xX)$ then K (w/ Chern conn) admits a c.c. section Ω, (In particular, a holomorphic trivialization.)

Similarly

Prop If $\mathrm{Hol}^0_x(\nabla) \subset SU(T_xX)$ then the Chern connection in K is flat.

This has a strong consequence:

Prop If $\mathrm{Hol}^0_x(\nabla) \subset SU(T_xX)$ then X is *Ricci-flat*.

Pf Recall *Ricci curvature* $r \in C^\infty(S^2(T^*X))$ $r(u,v) = \mathrm{tr}(w \mapsto F_\nabla(w,u)v)$
When X is Kähler, $r(u,v)$ is of type $(1,1)$ [i.e. $r(u,v) = 0$ if both $u,v \in T^{1,0}X$].

Define *Ricci form* $\mathrm{Ric} \in \Omega^2(X)$ by $\mathrm{Ric}(u,v) = r(I(u),v)$
[cf. $\omega(u,v) = g(I(u),v)$]

Then, $\mathrm{Ric} = i\,\mathrm{tr}(F_\nabla)$. (A direct computation Huyb p. 212)
But $\mathrm{tr}(F_\nabla)$ is the curvature of the induced ∇ on K. $\blacksquare$

How to actually find metrics of $SU(n)$ holonomy?

Lemma ω, ω' Kähler forms on X, with $\omega^n = e^f (\omega')^n$: then

$$Ric(\omega) = Ric(\omega') + i\partial\bar\partial f$$

Pf The connection ∇ induced on K is Chern for the norm $\|\alpha\|^2 = \frac{\alpha\wedge\bar\alpha}{\mathrm{vol}} \sim \frac{\alpha\wedge\bar\alpha}{\omega^n}$.

So changing the Kähler form from ω to ω' rescales the metric in K by a factor e^f, hence shifts curvature by $\partial\bar\partial f$. ∎

Thm (Yau) X compact Kähler, $\dim_{\mathbb{C}} X = n$, Kähler form ω

For any $f \in C^\infty(X, \mathbb{R})$ with

$$\int_X e^f \omega^n = \int_X \omega^n$$

there exists a unique Kähler metric on X w/ Kähler form ω', s.t. $[\omega'] = [\omega]$ and $(\omega')^n = e^f \omega^n$.

Pf Write $\omega' = \omega + i\partial\bar\partial\varphi$, then we have a nonlinear PDE for φ:

$$\det\left(\omega_{i\bar j} + \partial^2\varphi/\partial z_i \partial\bar z_j\right) = e^f \det(\omega_{i\bar j}) \qquad (\text{"Monge-Ampere"})$$

Show it has solution, collect Fields medal. ∎

Cor X compact Kähler, $\alpha \in H^2(X,\mathbb{R})$ a Kähler class.

$\beta \in H^{1,1}(X,\mathbb{R})$ closed, $[\beta] = c_1(X)$.

Then $\exists!$ Kähler form ω on X s.t.

1) $Ric(\omega) = 2\pi\beta$
2) $[\omega] = \alpha$

Pf Start with some ω_0, $[\omega_0] = \alpha$; then $[2\pi\beta] = [Ric(\omega_0)]$

so $2\pi\beta = Ric(\omega_0) + i\partial\bar\partial f$ ($\partial\bar\partial$-lemma).

Then $\exists\, \omega$ s.t. $\omega^n = e^{f+c}\omega_0^n$, $[\omega] = \alpha$.

That's the desired ω. ∎

Cor If X cpt Kähler, $c_1(X)=0$, $[\omega_0]=\alpha$
then $\exists$ a unique Ricci-flat Kähler metric on X with $[\omega]=\alpha$.

Pf Take $\beta=0$ in the above. ∎

Rk
- This is a completely inexplicit statement!
  One would like to be able to say something about what these Ricci-flat metrics actually look like.
- Kähler condition is essential: there are manifolds with $c_1(X)=0$ which have no Kähler metric at all!
- The existence of Ricci-flat Kähler metric is slightly weaker than $\mathrm{Hol}\subset SU(n)$: it is equivalent to $\mathrm{Hol}^\circ\subset SU(n)$. A Ricci-flat compact Kähler mfd does have a finite hol. covering which is a product of a torus and a space with $SU(n)$ holonomy. (Hard thm — see Besse p. 323)
- The thm gives a lot of examples: e.g. degree-$(n+1)$ hypersurfaces in $\mathbb{CP}^n$ admit Ricci-flat metrics.

Def A Calabi-Yau manifold is a mfd w/ holonomy contained in $SU(n)$.

Rk
- CY 1-folds: all diffeomorphic to T^2
  2-folds: " " " T^4 or K3
  3-folds: $O(1000)$ diffeo. types known, # not known to be finite
  ⋮
- Ricci-flat Kähler metrics generally come in (finite-dim^l) families.
  2 types of infinitesimal deformation:
  - change Kähler class — $h^{1,1}$-dim real family
  - change $\mathbb{C}$ structure — $h^{n-1,1}$-dim $\mathbb{C}$ family

[Why $h^{n-1,1}$? Inf^l def given by $H^1(X,T_{hol})$. Contracting w/ Ω gives $H^1(X,T_{hol})\simeq H^{n-1,1}(X)$.
Tian–Todorov: all infinitesimal $\mathbb{C}$ def actually integrate to actual def.]