Hyperkähler manifolds

Def A Riem. mfd (X, g, I_1, I_2, I_3) is <u>hyperkähler</u> if $I_{1,2,3}$ are complex structures on X <u>such that</u>

- $I_1 I_2 = I_3$ & cyclic perms. (this $\Rightarrow I_1 I_2 = -I_2 I_1$ etc)
- (X, g) is Kähler wrt all I_i

Let ω_i denote the corresp. 3 Kähler forms.

Prop If X is HK then $\Omega_1 = \omega_2 + i\omega_3$ is a <u>holomorphic symplectic form</u> with respect to complex structure I_1. [i.e. Ω_1 is a closed $(2,0)$ form inducing an isomorphism $T^{1,0}X \to (T^{1,0}X)^*$]

Pf
$$\begin{aligned}
\Omega_1(v,w) &= \omega_2(v,w) + i\omega_3(v,w) \\
&= g(I_2 v, w) + i g(I_3 v, w)
\end{aligned}$$
so
$$\begin{aligned}
\Omega_1(I_1 v, w) &= g(I_2 I_1 v, w) + i g(I_3 I_1 v, w) \\
&= -g(I_3 v, w) + i g(I_2 v, w) \\
&= i\Omega_1(v,w)
\end{aligned}$$
So Ω_1 is of type $(2,0)$. Nondegeneracy: for $v \in T^{1,0}$,

$$\Omega_1(v,\cdot) = 0 \Rightarrow \Omega_1(v+\bar{v}, \cdot) = 0 \Rightarrow \omega_2(v+\bar{v}, \cdot) = 0 \Rightarrow v + \bar{v} = 0 \Rightarrow v = 0 \quad \blacksquare$$

Cor If X is HK then

1) $\dim_{\mathbb{R}} X$ is a multiple of 4.
2) X is Ricci-flat.

Pf 1) existence of a hol. symplectic form means $T^{1,0}X$ is even-dimensional.
2) Ω^n is a cov. const. section of K. $\blacksquare$

$\mathbb{H}$ = quaternions

Simplest standard example is $\mathbb{H}^n$; each tangent space $\simeq \mathbb{H}^n$ acted on from (say) the right by quaternion multiplication. Flat metric.

Slightly fancier example: Calabi metric on $T^*\mathbb{CP}^1$. HK, complete.

Compact examples will of course be harder to come by!

A HK manifold has not just 3 complex structures, but a whole S^2 worth:

Prop If X is HK and $a_1^2+a_2^2+a_3^2=1$, $a_i \in \mathbb{R}$, then $I_{\vec{a}} = \sum_{j=1}^{3} a_j I_j$ is a complex structure on X.

Pf $I_{\vec{a}}^2 = -1$ directly.

For integrability, note $I_{\vec{a}}$ is cov. const. wrt connection ∇ so if X, Y are sections of $T^{1,0}$ then so are $\nabla_X Y$, $\nabla_Y X$.

But ∇ is torsion-free, so $\nabla_X Y - \nabla_Y X = [X,Y]$.

Hence $T^{1,0}$ is closed under $[\,,\,]$.

This is one of the characterizations of integrable cplx structures. ∎

So X naturally has an S^2 worth of complex structures. Viewing this S^2 as $\mathbb{CP}^1$, we could say X has $\mathbb{C}$ str $I^{(\zeta)}$.

First clue that this is a good idea: each $(X, I^{(\zeta)})$ is hol. symplectic, and the $\mathbb{C}$ symplectic form can be normalized as

$$\Omega(\zeta) = -\frac{i}{2\zeta}(\omega_1 + i\omega_2) + \omega_3 - \frac{i}{2}\zeta(\omega_1 - i\omega_2)$$

ie it depends holomorphically on ζ. (Although $J^{(\zeta)}$ doesn't!)

Let's explore this a little further, taking $X = \mathbb{R}^4$.

Determine an HK structure by taking 2 cplx coords, $\omega_1 + i\omega_2 = -2\, da\wedge db$,

$\omega_3 = i(da\wedge d\bar a + db\wedge d\bar b)$.

Then $\Omega(\zeta) = i\, dx(\zeta)\wedge dy(\zeta)$

$$x(\zeta) = \frac{a}{\zeta} - \bar b,$$
$$y(\zeta) = b + \bar a \zeta.$$

$\Omega = \frac{i}{\zeta} da\wedge db + i(da\wedge d\bar a + db\wedge d\bar b) + i\zeta\, d\bar a\wedge d\bar b$

$\Omega(\zeta)$ is $(2,0)$-form $\Longrightarrow$ the functions $x(\zeta), y(\zeta)$ are <u>holomorphic</u> wrt $I^{(\zeta)}$.

Another way of describing this situation: consider $Z = \mathbb{R}^4 \times \mathbb{CP}^1$.

- We have equipped Z with a <u>complex structure</u> (local hol. coords are $x(\zeta), y(\zeta), \zeta$)
- There is a <u>holomorphic projection</u>

$$\begin{array}{ccc} Z & & (x,y,\zeta) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \mathbb{CP}^1 & & \zeta \end{array}$$

- Points of $X = \mathbb{R}^4$ induce holomorphic <u>sections</u> of the projection, $\mathbb{CP}^1 \hookrightarrow Z$

$\zeta \mapsto (a, b, \zeta)$

- Ω determines a <u>global hol. section</u> of $\Omega^2_{Z/\mathbb{CP}^1} \otimes \pi^*(\mathcal{O}(2))$

ie Z is <u>fiberwise</u> <u>hol. symplectic</u>

Globally, $Z \simeq \mathcal{O}(1)\oplus\mathcal{O}(1) \longrightarrow \mathbb{CP}^1$ in this case. But more generally, such a Z can be constructed from any HK mfd X.

<u>Thm</u> Given Z such that:

- Z is a hol. fiber bundle $Z \xrightarrow{\pi} \mathbb{CP}^1$
- Z has a global hol. section Ω of $\Omega^2_{Z/\mathbb{CP}^1} \otimes \pi^*\mathcal{O}(2)$
- Z has a <u>real structure</u> ρ with $\rho^*\Omega = \overline{\Omega}$, covering <u>antipodal map</u>

[NB: antipodal map lifts to $\mathcal{O}(2) \to \overline{\mathcal{O}(2)}$, $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} -\bar b \\ \bar a \end{pmatrix}$]

- Z has a family X of <u>holomorphic sections</u> $s: \mathbb{CP}^1 \to Z$, invariant under ρ, with normal bundle $\simeq \mathcal{O}(1)^{\oplus 2n}$.

Then, X is naturally a pseudo-hyperkähler manifold of real dim $4n$.

Pf Sketch Idea (need deformation theory to make precise):

$(T_{\mathbb{C}}X)_s$ is the space of inf^l deformations of the section s,
i.e. it is $H^0[N(s(\mathbb{CP}^1))]$. Since $N \otimes \mathcal{O}(-1)$ is trivial, we have $H^0[N] \simeq H^0[N \otimes \mathcal{O}(-1)] \otimes H^0[\mathcal{O}(1)]$.

On $H^0[N \otimes \mathcal{O}(-1)]$ we have a skew pairing given by Ω.
On $H^0[\mathcal{O}(1)]$ there is also a canonical skew pairing, concretely

$$\langle a_1 + b_1\zeta,\ a_2 + b_2\zeta \rangle = a_1 b_2 - a_2 b_1 .$$

Combining these 2 gives the desired symmetric pairing on $H^0[N] = T_{\mathbb{C}}X$.
Keeping track of real structures, we see it induces a pairing on the real tangent space TX. This is the desired metric in X.

Then, have to show it's Kähler wrt all $\mathbb{C}$ str.
"Just" linear algebra [Hitchin-Karlhede-Lindstrom-Rocek]. ∎

Usefulness of this: so far, seems good mostly for deforming known (simple) HK structures — then the pb of positivity can be avoided, for sufficiently small deformations.

A second approach to existence of HK metrics:

Prop X compact Kähler w/ hol. symplectic form Ω, Hol_∇ acts irreducibly on T_xX:
Then for any Kähler class α, $\exists!$ hyperkähler metric on X
with $[\omega_1] = \alpha$, $\omega_2 + i\omega_3 = c\Omega$ for some $c \in \mathbb{R}$.

Pf $c_1(X) = 0$ since Ω^n gives hol. triv. of K.
By Yau's thm, $\exists!$ Ricci-flat Kähler metric g with $[\omega] = \alpha$.
Let ∇^* be formal adjoint of ∇. Weitzenbock: $\Delta_{\bar\partial} = \nabla^*\nabla$ (using $Ric=0$)
So, $0 = (\Omega, \Delta_{\bar\partial}\Omega) = (\nabla\Omega, \nabla\Omega)$, i.e. $\nabla\Omega = 0$.

Define $I_1 = I$, $\omega_1 = \omega$, $\omega_2 + i\omega_3 = \Omega$. Then I_2, I_3
determined by
$$\omega_2(\cdot,\cdot) = g(I_2\cdot,\cdot)$$
$$\omega_3(\cdot,\cdot) = g(I_3\cdot,\cdot)$$

Linear algebra exercise like we did before, using
$$(\omega_2 + i\omega_3)(I_1 v, \cdot) = (-\omega_3 + i\omega_2)(v, \cdot)$$
$$\begin{array}{ccc} \| & & \| \\ g(I_2 I_1 v, \cdot) + i g(I_3 I_1 v, \cdot) & & -g(I_3 v, \cdot) + i g(I_2 v, \cdot) \end{array}$$
shows
$$I_2 I_1 = -I_3$$
$$I_3 I_1 = I_2$$
But still need to get $I_2^2 = I_3^2 = -1$.

Since I_2 is skew-symmetric in an orthonormal basis, it can be conjugated into the form
$$\begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & \ddots \end{pmatrix}.$$
Then
$$I_2^2 = \begin{pmatrix} -a_1^2 & & & \\ & -a_1^2 & & \\ & & -a_2^2 & \\ & & & \ddots \end{pmatrix}$$

All eigenspaces are invariant under parallel transport, since $\nabla I_2 = 0$.
Thus irreducibility $\Rightarrow$ all eigenvalues of I_2^2 equal. Similarly I_3^2. And $I_2 I_1 = -I_3$
$\Rightarrow \det I_2 = \det I_3$.
Hence we can just rescale $I_j \to \lambda I_j$, $\omega_j \to \lambda\omega_j$
to arrange $I_2^2 = I_3^2 = -1$.

Still have $g(I_2\cdot,\cdot) = \omega_2(\cdot,\cdot)$
$g(I_3\cdot,\cdot) = \omega_3(\cdot,\cdot)$

And $\nabla I_2 = \nabla I_3 = 0$, $\nabla \omega_2 = \nabla \omega_3 = 0$ $\Rightarrow$ I_2, I_3 integrable, g Kähler for both $\blacksquare$

This is the missing ingredient we needed for the SYZ picture of K3: our original surface (X, g, I_1), fibered by complex tori, has <u>HK structure</u>.

Then (X, g, I_2) is <u>another</u> K3 surface, fibered by <u>special Lagrangian</u> tori. ✓

And Gross-Wilson showed that the <u>metric</u> behaves as predicted, in an appropriate "large complex structure" limit!