Riemannian Geometry: Exercise Set 2

Exercise 1

Solve Exercises 3-3, 3-4, 3-7, 3-8 of Lee.

Exercise 2

Let $(V, \langle , \rangle)$ be a real vector space with positive definite inner product. As discussed in class, V is canonically a Riemannian manifold.

1. Let $O(V)$ be the group of linear transformations of V preserving $\langle , \rangle$. Show that $O(V) \subset \mathrm{Isom}(V)$.
2. For $v \in V$ define the translation $\varphi_v : V \to V$ by $\varphi_v(w) = w + v$. Show that $\varphi_v \in \mathrm{Isom}(V)$, and the map $v \mapsto \varphi_v$ embeds V as an abelian subgroup of $\mathrm{Isom}(V)$.
3. Show that $\mathrm{Isom}(V)$ contains the semidirect product of V and $O(V)$, with $O(V)$ acting on V in the obvious way. (Later we will prove that $\mathrm{Isom}(V)$ *equals* this semidirect product.)

Exercise 3

1. Let $(M, g) \subset (\tilde{M}, \tilde{g})$ be an embedded Riemannian submanifold. Suppose $h \in \mathrm{Isom}(\tilde{M}, \tilde{g})$ restricts to a map $M \to M$. Show that $h \in \mathrm{Isom}(M, g)$.
2. Let $(V, \langle , \rangle)$ be a real vector space with positive definite inner product. Let $S(V) = \{v \in V : \|v\| = 1\}$. Show that $O(V) \subset \mathrm{Isom}(S(V))$.

Exercise 4

Let (M, g) be a Riemannian manifold. Let $\langle , \rangle$ denote the inner product on any $T^k_l(M)$ induced by g.

1. Suppose $\eta, \omega \in T^1(M)$. Show that $\langle \eta^\sharp, \omega^\sharp \rangle = \langle \eta, \omega \rangle$.

Exercise 5

Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$.

1. Show that $\mathrm{grad}\, f$ points in the direction of fastest increase of the function f (part of the exercise is to figure out precisely what this sentence should mean).