Riemannian metrics

M smooth connected manifold, of dimension n

Def A Riemannian metric on M is a smooth, symmetric, positive definite $g \in T^2(M)$

$\left[g(p) \in (T_p^* M)^{\otimes 2} \text{ equiv to } g(p) : (T_p M)^{\otimes 2} \longrightarrow \mathbb{R}, \text{ with } X \neq 0 \Rightarrow [g(p)](X,X) > 0 \right]$

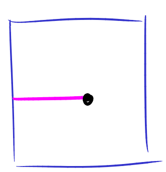
In coordinate frame, $g = g_{ij}(x)\, dx^i \otimes dx^j$ with $g_{ij} = g_{ji}$

or, introducing symmetric product $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$,

$$g = g_{ij}(x)\, dx^i dx^j$$

Ex Canonical metric on $\mathbb{R}^n$: $g_{can} = \sum_{i=1}^{n} (dx^i)^2$ so $(g_{can})_{ij} = \delta_{ij}$ $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

On $\mathbb{R}^2$, in polar coords $x^1 = r\cos\theta$, $x^2 = r\sin\theta$



$$\begin{aligned} g_{can} &= (dx^1)^2 + (dx^2)^2 \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, dr)^2 \\ &= dr^2 + r^2 d\theta^2 \end{aligned} \qquad \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

[NB, g appears to become degenerate at $r=0$! But miraculously it extends...]

Ex $(V, \langle , \rangle)$ any real inner product space: $T_p V \simeq V$ so $g(X,Y) = \langle X, Y \rangle$ gives a metric on V

Ex $(\tilde{M}, \tilde{g})$ Riem mfd, $\iota : M \longrightarrow \tilde{M}$ immersion

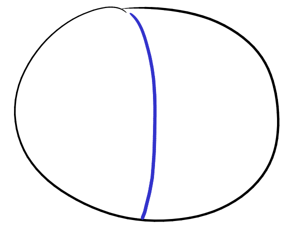
then $g = \iota^* \tilde{g}$ is a Riem metric on M ("induced metric")

[**Ex** standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$ induces "round metric" on S^n]

Concretely, if u^i are local coordinates on a patch of $\tilde{M}$ and ι is given by $x^{i'}(u^1, \ldots, u^m)$,

$$g_{ij} = \tilde{g}_{i'j'} \frac{\partial x^{i'}}{\partial u^i} \frac{\partial x^{j'}}{\partial u^j}$$

Ex $M = S^2$: local coords $u^1 = \theta$, $u^2 = \varphi$
$[0 < \theta < \pi,\ 0 < \varphi < 2\pi]$



$$\iota: S^2 \longrightarrow \mathbb{R}^3$$
$$(\theta, \varphi) \longmapsto (R\cos\theta, R\sin\theta\sin\varphi, R\sin\theta\cos\varphi)$$

$$g = \left(d(R\cos\theta)^2\right) + \left(d(R\sin\theta\sin\varphi)^2\right) + \left(d(R\sin\theta\cos\varphi)^2\right)$$
$$= R^2 \cdot \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$g_{\theta\theta} = R^2 \qquad g_{\varphi\varphi} = R^2\sin^2\theta \qquad g_{\theta\varphi} = 0$$

Ex If (M_1, g_1) and (M_2, g_2) are Riem mfds
then $T_{(x_1,x_2)}(M_1 \times M_2) = T_{x_1}M_1 \oplus T_{x_2}M_2$
and we define a metric g on $M_1 \times M_2$ by $g(X_1 + X_2, X_1' + X_2') = g_1(X_1, X_1') + g_2(X_2, X_2')$
ie "$g = g_1 \oplus g_2$"

$$g = \left(\begin{array}{c|c} g_1 & \\ \hline & g_2 \end{array}\right)$$

Def If (M, g) and $(\tilde{M}, \tilde{g})$ are Riem mfds, an isometry from M to $\tilde{M}$
is $\varphi: M \longrightarrow \tilde{M}$ diffeomorphism with $\varphi^*\tilde{g} = g$

Def $\mathrm{Isom}(M) = \{\varphi: M \longrightarrow M \text{ isometry}\}$

$\mathrm{Isom}(M)$ is a group. (actually Lie group wrt compact-open topology [Myers-Steenrod])

Ex $(V, \langle , \rangle)$ inner product space: $\mathrm{Isom}(V) = V \rtimes O(V)$
[Easy to see $\supset$
Later we'll prove $\subset$]