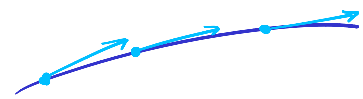
Connections

Want a useful notion of "straight line" in a Riem mfd (M,g).

Fact: $\gamma : [0,T] \to \mathbb{R}^n$ is a straight line iff $\exists\, \tau : [0,T] \to [0,T]$

s.t. $\tilde{\gamma} = \gamma \circ \tau$ has $\frac{d}{d\tau}(\dot{\tilde{\gamma}}) = 0$, ie $\dot{\tilde{\gamma}} = \text{const}$

How to make sense of this eq. on a general (M,g)? It makes no sense *a priori* since it involves comparing vectors in different tangent spaces.



Nevertheless we want to define it.

Intuition: suppose $\iota : M \hookrightarrow \mathbb{R}^n$ submanifold, $\gamma : [0,T] \to M$, and $X \in \Gamma(\gamma^* TM)$

ie $X(t) \in T_{\gamma(t)} M$. Then have $d\iota : T_p M \to T_p \mathbb{R}^n \simeq \mathbb{R}^n$

$\pi : \mathbb{R}^n \simeq T_p \mathbb{R}^n \to T_p M$ orth. proj.

Then we could define $D_t X = \pi\left(\frac{d}{dt} X\right)$.

Def M smooth mfd, $E \to M$ real vector bundle.

A connection in E is, for each open set $U \subset M$, a map

$$\nabla^{(U)} : \mathcal{E}(E|_U) \longrightarrow \mathcal{E}(E|_U \otimes T^* U)$$

- compatible with restriction of U i.e. for $U' \subset U$, $\nabla^{(U)}(s)\big|_{U'} = \nabla^{(U')}(s|_U)$
- $\mathbb{R}$-linear i.e. $\nabla(cs + c's') = c\nabla(s) + c'\nabla(s')$

(i.e. ∇ is a map of sheaves of $\mathbb{R}$-modules)

- Leibniz i.e. $\nabla(fs) = (df)\, s + f \nabla s$

Notation: usually write $\nabla_X s$ for $\iota_X \nabla s$. ("covariant derivative in the direction X")

In local coordinates we write $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$.

Suppose we fix a local basis of sections $\{e_a\}_{a=1,\dots,r}$ for E (but not a local coordinate.)

Define "connection coeff." $A_a^b \in T^1(M)$ by $\nabla e_a = A_a^b e_b$

Expanding a general section as $s = s^a e_a$ we have

$$\nabla s = (ds^a) e_a + s^a \nabla e_a = (ds^a) e_a + s^a A_a^b e_b = (ds^a + A_b^a s^b) e_a$$

i.e. $(\nabla s)^a = ds^a + A_b^a s^b$

It's tempting to organize A_b^a into $A = A_b^a e_a \otimes e^b \in T^1 \otimes \mathrm{End}\, E$

But, consider what happens under a change of basis, $e_{a'} = C_{a'}^a e_a$

We'll then have
$$\nabla e_{a'} = \nabla(C_{a'}^a e_a) = dC_{a'}^a e_a + C_{a'}^a A_a^b e_b$$
$$= (C^{-1})_a^{b'} dC_{a'}^a e_{b'} + C_{a'}^a A_a^b (C^{-1})_b^{b'} e_{b'}$$

ie $A_{a'}^{b'} = (C^{-1})_b^{b'} A_a^b C_{a'}^a + (C^{-1})_a^{b'} dC_{a'}^a$

The second term reflects the fact that a connection is not a global section of any vector bundle over M.

However, what is true is that the difference of two connections is a global $\mathrm{End}(E)$-valued 1-form: ie

$$(\nabla - \tilde{\nabla}) s = \omega \cdot s \qquad \omega \in \mathcal{E}(T^* \otimes \mathrm{End}(E))$$

[One way to understand this: A and $\tilde{A}$ both transform by the same inhom. term, so $A - \tilde{A}$ is indep. of basis. Another way: $\nabla - \tilde{\nabla}$ is linear over C^∞ functions, by Leibniz rule.]

Lemma M smooth, E v.b.

$\implies$ the space of connections on E is an affine space modeled on $\mathcal{E}(T^1 \otimes \mathrm{End}\, E)$.

Pf Just need to show a connection exists: use partition of unity.

Given $(\tilde{M}, E, \nabla)$ and $\varphi: M \to \tilde{M}$, there is <u>pullback connection</u> $\varphi^*\nabla$ on φ^*E, as follows:

Fix a local basis $\{e_a\}$ for E, then wrt the basis $\{\varphi^* e_a\}$ for $\varphi^* E$,
define $\varphi^*\nabla$ by fixing its connection coeff. to be $\varphi^* A^a_b$.
Then check that $\varphi^*\nabla$ is indep. of the chosen basis (basically $\varphi^*(C^{-1}dC) = \varphi^*(C^{-1})\, d\,\varphi^*(C)$).

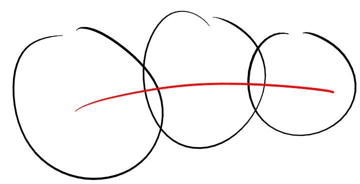
[$\varphi^*\nabla$ can also be characterized by the confusing equation:

$$(\varphi^*\nabla)_X(\varphi^* s) = \varphi^*\left(\nabla_{\varphi_*(X)} s\right)$$
]

<u>Lemma</u> Fix $\gamma: [0,1] \to M$ and $s_0 \in E_{\gamma(0)}$.

Then $\exists!\ s \in \mathcal{E}(\gamma^* E)$ s.t. $(\gamma^*\nabla)_{\partial/\partial t}\, s = 0$, $s(0) = s_0$.

<u>Pf</u> Show that the interval on which s exists is:

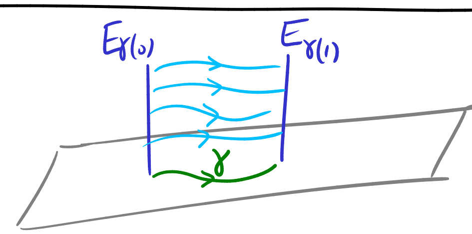


<u>open</u> — by theory of linear 1^{st}-order ODE in a coord. chart

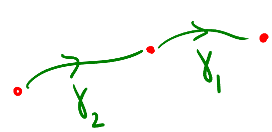
[$\dfrac{ds^a}{dt} = -c^a_b(t)\, s^b$, where $(c^a_b\, dt)$ are the connection forms for $\gamma^*\nabla$
or explicitly, $c^a_b = [A^a_b(\dot{\gamma})]$]

<u>closed</u> — by taking a chart containing the limit point

<u>Def</u> (Parallel transport) $P_{\nabla,\gamma}(s_0) = s(1)$.



<u>Lemma</u>
- $P_{\nabla,\gamma}$ is an isomorphism $E_{\gamma(0)} \xrightarrow{\sim} E_{\gamma(1)}$.
- If $\gamma = \gamma_1\gamma_2$ then $P_{\nabla,\gamma_1} \circ P_{\nabla,\gamma_2} = P_{\nabla,\gamma}$.



<u>Operations on connections</u>

- Given ∇_1, ∇_2 in E_1, E_2 get ∇ on $E_1 \oplus E_2$ by $\nabla(s_1, s_2) = (\nabla s_1, \nabla s_2)$ $\quad A = \left(\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right)$

  ∇ on $E_1 \otimes E_2$ by $\nabla(s_1 \otimes s_2) = \nabla s_1 \otimes s_2 + s_1 \otimes \nabla s_2$
- Given ∇ on E get ∇ on E^* by $d(\omega \cdot s) = (\nabla\omega)\cdot s + \omega(\nabla s)$