Andy class.

$U \subset \mathbb{A}^n$ open.
$x^1, \ldots, x^n$

$g_{ij} : U \to \mathbb{R}$ metric : $g = g_{ij}\, dx^i \otimes dx^j$

$x : [0,T] \longrightarrow U$ smooth. $x(t) = (x^1(t), \ldots, x^n(t))$

$$L(x) = \int_0^T dt \sqrt{g_{ij}(x(t))\, \dot{x}^i(t)\, \dot{x}^j(t)}$$

Rmk: $L(x)$ is reparametrization-invariant: $[0,T'] \xrightarrow[\cong]{\phi} [0,T]$, $t' \mapsto t$, monotonic increasing.

$$= \int_0^{T'} dt' \sqrt{g_{ij}(x(t(t'))\frac{d}{dt'}x^i(t(t'))\frac{d}{dt'}x^j(t(t'))}$$

Prop: If $\dot{\gamma}(t) \neq 0$ for all t, then exists a unique ϕ s.t. $\left\|\frac{d}{dt'}\gamma(t(t'))\right\| = 1$.

Assume so for original parametrization; then $L(\gamma) = T$.

Now differentiate length.

$\Gamma : [0,T] \times (-\epsilon, \epsilon) \longrightarrow U$ smooth. $\gamma_s(t) = \Gamma(t,s)$

$\Gamma(t,0) = \gamma(t)$ parametrized by arc length.

$\left.\frac{d}{ds}\right|_{s=0} L(\gamma_s)$ is variation of arc length.

Note $[\partial_t, \partial_s] = 0$: mixed partials commute.

$$\frac{d}{ds}\Big|_{s=0} L(x_s) = \partial_s\Big|_{s=0} \int_0^T dt \sqrt{g_{ij}(X(t,s))\,\partial_t x^i(t,s)\,\partial_t x^j(t,s)}$$

$$= \int_0^T dt \; \frac{\partial_s g_{ij}(X)\,\partial_t x^i\,\partial_t x^j + g_{ij}(X)\,\partial_s\partial_t x^i\,\partial_t x^j + g_{ij}(X)\,\partial_t x^i\,\partial_s\partial_t x^j}{2\sqrt{g_{ij}(X)\,\partial_t x^i\,\partial_t x^j}}\Bigg|_{s=0} \quad [\text{uniformly differentiable}].$$

$$= \int_0^T dt \; \frac{1}{2}\frac{\partial g_{ij}(x)}{\partial x^k}\partial_s x^k\,\partial_t x^i\,\partial_t x^j + 2 g_{ij}(x)\,\partial_s\partial_t x^i\,\partial_t x^j$$

$$= \int_0^T dt \Big[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\,\partial_t x^i\,\partial_t x^j\,\partial_s x^k + \partial_t\{g_{ij}(x)\,\partial_s x^i\,\partial_t x^j\} - \frac{\partial g_{ij}}{\partial x^k}\,\partial_t x^k\,\partial_t x^j\,\partial_s x^i - g_{ij}(x)\,\partial_t^2 x^j\,\partial_s x^i\Big]$$

$$= \int_0^T dt \Big[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\,\dot{x}^i\dot{x}^j - \frac{\partial g_{kj}}{\partial x^i}\,\dot{x}^i\dot{x}^j - g_{kj}\,\ddot{x}^j\Big]\partial_s x^k$$

$$+ \; g_{ij}\,\dot{x}^i\,\partial_s x^j\Big|_0^T$$

If to be zero,

$$g_{kj}\ddot{x}^j + \frac{\partial g_{kj}}{\partial x^i}\dot{x}^i\dot{x}^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = 0 \qquad \text{for all } k.$$

Rewrite:

$$\left[g_{kj}\partial_t + \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\dot{x}^i \right]\dot{x}^j = 0 \qquad \forall k$$

or

$$\partial_t \dot{x}^l + \frac{1}{2}g^{kl}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\dot{x}^i\dot{x}^j = 0 \qquad \forall l$$

Set

$$\Gamma^l_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) : U \longrightarrow \mathbb{R}$$

Christoffel (1869)

So

$$\boxed{\partial_t \dot{x}^l + \Gamma^l_{ij}\dot{x}^i\dot{x}^j = 0 \qquad \forall l} \qquad (*)$$

Define covariant derivative operator on $E = U \times \mathbb{R}^n$ by

$$\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j} = \Gamma^l_{ij}\frac{\partial}{\partial x^l}, \qquad \Gamma^l_{ij} : U \longrightarrow \mathbb{R}.$$

So $(*)$ is

$$\nabla_{\dot{x}^i\partial/\partial x^i}\left(\dot{x}^j\frac{\partial}{\partial x^j}\right) = \dot{x}^i\frac{\partial \dot{x}^j}{\partial x^i}\frac{\partial}{\partial x^j} + \dot{x}^i\dot{x}^j\Gamma^l_{ij}\frac{\partial}{\partial x^l}$$

$$= \left(\partial_t \dot{x}^l + \Gamma^l_{ij}\dot{x}^i\dot{x}^j\right)\frac{\partial}{\partial x^l}$$

Explain that computing for $[0,T] \to \mathbb{R}^n$, so pullback ∇

so if $\dot{x} : [0,T] \to \mathbb{R}^n$, then $\boxed{\nabla_{\dot{x}} \dot{x} = 0}$ geodesic eqn.

Interpretation: · Zero acceleration

· $\dot{x}$ only defined along x ~ best interpretation on
$$\begin{array}{c} x^*E \\ \downarrow \uparrow \dot{x} \\ [0,T] \end{array}$$

pullback covariant derivative.

Intrinsic approach: ① $\begin{array}{c} E, \langle\,\rangle \text{ metric.} \\ \downarrow \\ M \end{array}$

∇ ~~compat~~ orthogonal if

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle \quad \text{for all sections } s_1, s_2.$$

Equivalently:
$$g : E \otimes E \to \underline{\mathbb{R}}$$
$$\nabla g = 0.$$

② $\begin{array}{c} TM \\ \downarrow \\ M \end{array}$, ∇ covariant derivative

Defn: Torsion $T(\xi_1, \xi_2) = \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]$

$T : \Lambda^2 TM \to TM$. tensor.

Thm (Levi-Civita) (Ricci, his teacher) (1917): M Riemannian. $\exists!\ \nabla$ orthogonal, torsion-free.

Pf: Compute $\langle \nabla_{\xi_1}\xi_2, \xi_3\rangle$.

Mysterious fundamental part of Riemannian geometry!

In 1900 he and Ricci-Curbastro published the theory of tensors in *Méthodes de calcul différentiel absolu et leurs applications*, which Albert Einstein used as a resource to master the tensor calculus, a critical tool in Einstein's development of the theory of general relativity. Levi-Civita's series of papers on the problem of a static gravitational field were also discussed in his 1915–1917 correspondence with Einstein. The correspondence was initiated by Levi-Civita, as he found mathematical errors in Einstein's use of tensor calculus to explain theory of relativity. Levi-Civita methodically kept all of Einstein's replies to him, and even though Einstein hadn't kept Levi-Civita's, the entire correspondence could be re-constructed from Levi-Civita's archive. It's evident from these letters that, after numerous letters, the two men had grown to respect each other. In one of the letters, regarding Levi-Civita's new work, Einstein wrote "I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot". In 1933 Levi-Civita contributed to Paul Dirac's equations in quantum mechanics as well.[6]

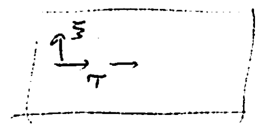
In coord system derive $\nabla_{\partial/\partial x^i} \partial/\partial x^j$.

Finally, revisit computation: $\gamma : [0,T] \longrightarrow M$

$$\Gamma : \underset{\overset{\shortparallel}{P}}{[0,T] \times (-\epsilon,\epsilon)} \longrightarrow M$$

Compute on P. Sections

$$\tau = \frac{\partial \Gamma}{\partial t}, \qquad \xi = \frac{\partial \Gamma}{\partial s}$$



of $\Gamma^* TM \longrightarrow P$. So

$$L = \int_0^T dt \, \langle \tau, \tau \rangle^{1/2}$$ (evaluated at s).

$$\partial_s L = \partial_s \int_0^T dt \, \langle \tau, \tau \rangle^{1/2} \Big|_{s=0}$$

$$= \int_0^T dt \, \partial_s \langle \tau, \tau \rangle^{1/2} \Big|_{s=0}$$

$$= \int_0^T dt \, \frac{\langle \nabla_{\partial_s} \tau, \tau \rangle}{\langle \tau, \tau \rangle^{1/2}} \Big|_{s=0}$$ orthogonal

$$= \int_0^T dt \, \langle \nabla_{\partial_t} \xi, \tau \rangle \Big|_{s=0}$$ torsionfree, $\langle \tau, \tau \rangle = 1$ at $s=0$

$$= \int_0^T dt \left\{ \partial_t \langle \xi, \tau \rangle - \langle \xi, \nabla_{\partial_t} \tau \rangle \right\} \Big|_{s=0}$$ orthogonal

$$= \langle \xi, \tau \rangle \Big|_0^T - \int_0^T \langle \xi, \nabla_{\partial_t} \tau \rangle \, dt .$$

See again geodesic equation $\nabla_{\partial_t} \tau = 0$.