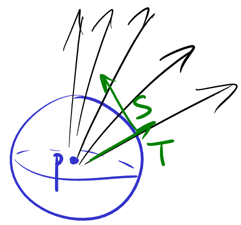
In normal coordinates at p, geodesics thru p ("radial geodesics") are simple: just $[\gamma_v(t)]^i = tv^i$
(or: $\gamma_v(t) = tv$)

<u>Def</u> <u>Geodesic ε-sphere</u> around p = locus $\{r = \varepsilon\}$ in normal coords. around p.
<u>Geodesic ε-ball</u> around p = locus $\{r \leq \varepsilon\}$ in normal coords. around p.

<u>Thm</u> (Gauss lemma): In normal coords around p, $\partial/\partial r$ is <u>orthogonal</u> to geodesic spheres around p.

<u>Pf</u> Consider family of geodesics thru the origin:



$$T = \gamma_*\left(\frac{\partial}{\partial t}\right) \qquad S = \gamma_*\left(\frac{\partial}{\partial s}\right)$$

$$\gamma(s,t) = \exp(t\sigma(s))$$

for some $\sigma: (\varepsilon, -\varepsilon) \longrightarrow$ (unit sphere in T_pM)

First note $\nabla_T S = \nabla_S T$ (because $[S,T] = 0$ and ∇ is torsion-free)

$$\frac{d}{dt}\langle S, T\rangle = \langle \nabla_T S, T\rangle + \cancel{\langle S, \nabla_T T\rangle}^{\,0}$$
$$= \langle \nabla_S T, T\rangle$$
$$= \frac{1}{2}\frac{\partial}{\partial s}\|T\|^2 = \frac{1}{2}\frac{\partial}{\partial s}(1) = 0$$

and $S(t=0) = 0$ [since $\gamma(s,0) = p$ independent of s]

so $\langle S, T\rangle = 0$ which is what we wanted to prove.

<u>Cor</u> $\operatorname{grad}(r) = \dfrac{\partial}{\partial r}$.
<u>Pf</u> direct computation

<u>Prop</u> Say $p \in M$ and $q \in$ geodesic ball around p.
Then radial geodesic from p to q is unique minimizing curve from p to q in M.

<u>Pf</u> For any path γ from p to q in the ball, $L(\gamma) = \int \|\dot\gamma\|\,dt = \int \sqrt{\|\dot\gamma_r\|^2 + \|\dot\gamma_t\|^2}\,dt$ $\quad (\dot\gamma = \dot\gamma_r + \dot\gamma_t)$

$$\geq \int \|\dot\gamma_r\|\,dt$$

radial $\quad$ tangential

then writing $\dot{\gamma}_r = \alpha \frac{\partial}{\partial r}$,

$$= \int |\alpha| \, dt$$
$$\geq \int \alpha \, dt$$
$$= \int \frac{d(r(\gamma(t)))}{dt} dt = r(q)$$

So, to minimize this, need to arrange $\dot{\gamma}_t = 0$, i.e. γ is (a reparameterization of) the radial geodesic.

(Taking paths which can go <u>outside</u> the ball clearly won't help.)

<u>Cor</u> Say $p \in M$ and $q \in$ geodesic ball around p.
Then $d(p,q) = r(q)$.

So a geodesic ε-ball is also a metric ε-ball.

<u>Thm</u> γ geodesic $\implies$ γ can be covered by patches W s.t. for $\gamma(t_1), \gamma(t_2) \in W$, γ is minimizing between $\gamma(t_1)$ and $\gamma(t_2)$.

[<u>Rk</u> We can't ask for more: e.g. a great circle on S^2 is <u>not</u> even a local minimum of length]

<u>Pf</u> Cover γ by nbhds W s.t. W is contained in a geodesic ball around any $p \in W$. [see Lemma below] Then, for any $\gamma(t_1)$ and $\gamma(t_2)$ in W, $\gamma|_{[t_1,t_2]}$ is <u>radial</u> geodesic in normal coords around $\gamma(t_1)$, hence is minimizing between $\gamma(t_1)$ and $\gamma(t_2)$.