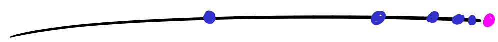
Def (M,g) is <u>geodesically complete</u> if all geodesics in M are defined for all $t \in \mathbb{R}$.

Ex $\mathbb{R}^n$ is geodesically complete. $\mathbb{R}^n \setminus U$ for U nonempty is not geodesically complete.

Thm (Hopf–Rinow) (M,g) is geodesically complete $\iff$ (M,d) is complete metric space.

Pf ($\Rightarrow$) Say (M,d) complete but (M,g) geod. incomplete.



Fix a geodesic γ which ends at time t_*.

Then take a sequence $t_n \to t_*$. $\{\gamma(t_n)\}$ is Cauchy sequence, since $\gamma(t_n)$ so converges to some p.

But then look at a uniformly normal δ-neighborhood W of p.

W contains some $\gamma(t_n)$ with $t_n > t_* - \delta/2$. Then look at geodesic beginning from t_n w/ speed $\dot\gamma(t_n)$.

$\Rightarrow$ can extend γ to at least $t_n + \delta > t_* + \delta/2$. ※

($\Leftarrow$) <u>First show</u>: if $T_pM \subset D$, i.e. exp is defined on all of T_pM, then $\forall q$, $\exists$ <u>length-minimizing geodesic from p to q.</u>

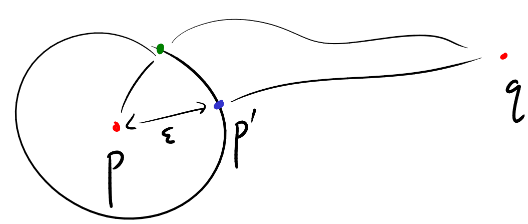
Take a geodesic ε-sphere $S_\varepsilon(p)$. Let p' be a point of $S_\varepsilon(p)$ minimizing $d(p',q)$.

Let γ be the radial geodesic from p to p', <u>extended</u> for all time.

Now, define $T = \sup\{t : \gamma|_{[0,t]}$ is length-minimizing and has $d(p,\gamma(t)) + d(\gamma(t),q) = d(p,q)\}$

<u>First, show $T \geq \varepsilon$.</u> Indeed $d(p,q) = d(p',q) + \varepsilon$, and $\gamma|_{[0,\varepsilon]}$ is radial $\Rightarrow$ it's length-min.

$$\left[\begin{array}{ll} d(p,q) \leq d(p',q) + \varepsilon & \triangle \text{ ineq.} \\ d(p,q) \geq d(p',q) + \varepsilon & \text{b/c any path from } p \text{ to } q \\ & \text{takes} \geq \varepsilon \text{ to reach } S_\varepsilon(p) \\ & \text{and from there it takes} \geq d(p',q). \end{array}\right]$$



<u>Next, show T is actually attained</u> i.e. can replace "sup" by "max" in def of T. This just uses continuity of d.

<u>Next, suppose $T < d(p,q)$.</u> Let $p' = \gamma(T)$.

Let q' be a minimizer of $d(q',q)$ on a geodesic sphere $S_\varepsilon(p')$.

Then, by same argument as before, $d(q,q') + \varepsilon = d(p',q)$

but also $d(p,p') + d(p',q) = d(p,q)$ by def of T.

Combine these $\rightsquigarrow$ $d(p,q) - d(q,q') = d(p,p') + \varepsilon$.

Then, Δ ineq $\Rightarrow$ $d(p,q') \geq d(p,p') + \varepsilon$.

Now let τ be the radial geodesic from p' to q'.

The concat. $\gamma|_{[0,T]}\tau$ has length $= d(p,p') + \varepsilon$.

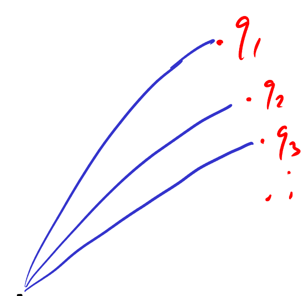
Thus, $\gamma|_{[0,T]}\tau$ minimizes distance from p to q'!

Thus, (using variation formula) $\gamma|_{[0,T]}\tau$ cannot have a corner, so $\gamma|_{[0,T]}\tau$ is a geodesic, i.e. coincides with $\gamma|_{[0,T+\varepsilon]}$. But

$$\begin{aligned} d(p,q) &= d(p,p') + d(p',q) \\ &= d(p,p') + (d(q',q) + \varepsilon) \\ &= (d(p,p') + \varepsilon) + d(q',q) = d(p,q') + d(q',q) \end{aligned}$$

This contradicts maximality of T.

So $T \geq d(p,q)$. But then $\gamma|_{[0,d(p,q)]}$ is the desired length-min. geodesic from p to q.



Now, suppose $\{q_n\}$ Cauchy sequence in M.

Take minimizing geodesics from p which end at q_n,

$$q_n = \exp(t_n v_n), \quad \|v_n\| = 1.$$

$t_n = d(p,q_n)$, so $\{t_n\}$ is bdd, so $\{t_n v_n\}$ is bdd, so $\{t_n v_n\}$ has convergent subseq, $t_{n_i} v_{n_i} \to v$. Then $q_{n_i} \to \exp(v)$ since exp is cts.

But then since $\{q_n\}$ is Cauchy, $q_n \to \exp(v)$.