Riemann curvature

Levi-Civita connection ∇ in TM

because ∇ is orthogonal, (see exercises)

Denote its curvature by $R = F_\nabla \in \mathcal{E}(\Lambda^2 T^*M \otimes \mathfrak{o}(TM)) \subset \mathcal{E}(\Lambda^2 T^*M \otimes \operatorname{End} TM)$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Components: $R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l$

Lower the last index to get Rm:

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle = -Rm(X,Y,W,Z)$$

$$Rm(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl}$$

Symmetries: 1) $Rm(X,Y,Z,W) = -Rm(Y,X,Z,W) = -Rm(X,Y,W,Z)$

2) $Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0$

3) $Rm(X,Y,Z,W) = Rm(Z,W,X,Y)$

since R is $\mathfrak{o}(TM)$-valued

Pf 1) $Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle = -Rm(X,Y,W,Z)$

2) Let $\mathfrak{S}T(X,Y,Z) = T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y)$ for any T.

Then
$$\begin{aligned}
\mathfrak{S}R(X,Y)Z &= \mathfrak{S}\nabla_X\nabla_Y Z - \mathfrak{S}\nabla_Y\nabla_X Z - \mathfrak{S}\nabla_{[X,Y]}Z \\
&= \mathfrak{S}\nabla_Z\nabla_X Y - \mathfrak{S}\nabla_Z\nabla_Y X - \mathfrak{S}\nabla_{[X,Y]}Z \\
&= \mathfrak{S}\nabla_Z(\nabla_X Y - \nabla_Y X) - \mathfrak{S}\nabla_{[X,Y]}Z \\
&= \mathfrak{S}\left(\nabla_Z [X,Y] - \nabla_{[X,Y]}Z\right) \\
&= \mathfrak{S}[Z,[X,Y]] \\
&= 0 \quad \text{by Jacobi}
\end{aligned}$$

since ∇ is torsion-free

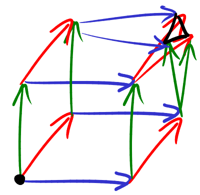
3) follows from 1), 2): $R(X,Y,Z,W) = -R(Z,X,Y,W) - R(Y,Z,X,W)$

$$= R(Z,X,W,Y) + R(Y,Z,W,X)$$

$$= -R(X,W,Z,Y) - R(W,Z,X,Y) - R(Z,W,Y,X) - R(W,Y,Z,X)$$
$$= 2R(Z,W,X,Y) + R(X,W,Y,Z) + R(W,Y,X,Z)$$
$$= 2R(Z,W,X,Y) - R(Y,X,W,Z)$$
$$= 2R(Z,W,X,Y) - R(X,Y,Z,W)$$

so $\quad R(X,Y,Z,W) = R(Z,W,X,Y)$

Rk 1) Geometric interpretation of 2):



2) Every Rm with these symmetries can be realized: if C_{ijkl} obeys 1), 2) then

$$g_{ik} = \delta_{ik} - \tfrac{1}{6}(C_{ijlk} + C_{iljk})\, x^j x^l = \delta_{ik} - \tfrac{1}{3} C_{ijkl}\, x^j x^l \qquad (\star)$$

has $Rm(x=0)_{ijkl} = C_{ijkl}$. [We'll prove it below.]

3) If M is a Lie gp G with a bi-inv't metric, then $\mathfrak{g} \simeq T_e G$ and $R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$ where $[\,,]$ means the bracket in $\mathfrak{g}$ (or the bracket of left-inv't vector fields)

So here the symmetry 2) becomes literally the Jacobi identity!

Def d_∇ acts on 2-forms by
$$d_\nabla G(X,Y,Z) = \nabla_X G(Y,Z) + \nabla_Y G(Z,X) + \nabla_Z G(X,Y) - G([X,Y],Z) - G([Y,Z],X) - G([Z,X],Y)$$

(Differential) Bianchi identity

For any connection one has $d_\nabla F_\nabla = 0$.

(cf. abelian case, where locally $F_\nabla = dA$ so $dF_\nabla = d^2 A = 0$)

Pf Recall if $\nabla = \nabla' + A$, ∇' flat, $A \in \mathcal{E}(T^* \otimes \operatorname{End} E)$

$$F_\nabla = d_{\nabla'} A + A \wedge A \in \mathcal{E}(\operatorname{End} E)$$

and $d_\nabla G = d_{\nabla'} G + A \wedge G - G \wedge A$ for $G \in \mathcal{E}(\Lambda^2 T^* \otimes \operatorname{End} E)$ [Exercise]

$$d_\nabla F_\nabla = d_{\nabla'}(d_{\nabla'} A + A\wedge A) + A \wedge (d_{\nabla'} A + A \wedge A) - (d_{\nabla'} A + A \wedge A)\wedge A$$
$$= d_{\nabla'} A \wedge A - A \wedge d_{\nabla'} A + A \wedge d_{\nabla'} A - d_{\nabla'} A \wedge A + A\wedge A \wedge A - A \wedge A \wedge A$$
$$= 0$$

In particular this applies to R. Moreover, using the torsion-free condition it can be rewritten purely in terms of the tensor ∇R:

Prop $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$

Pf $\mathcal{C}\nabla_X(R(Y,Z)) = \mathcal{C}[(\nabla_X R)(Y,Z) + R(\nabla_X Y, Z) + R(Y, \nabla_X Z)]$

$$= \mathcal{C}[(\nabla_X R)(Y,Z) + R(\nabla_X Y, Z) - R(\nabla_Y X, Z)]$$

$$= \mathcal{C}[(\nabla_X R)(Y,Z) + R([X,Y],Z)]$$

so $\mathcal{C}[(\nabla_X R)(Y,Z)] = \mathcal{C}[\nabla_X(R(Y,Z)) - R([X,Y],Z)] = 0$

Prop $\varphi: M \to \tilde{M}$ isometry: $\varphi^* \tilde{R} = R$.

Prop [Riemann] Say $R = 0$.
Then any $p \in M$ has a nbhd isometric to an open subset of $\mathbb{R}^n$.

Pf Fix $p \in M$ and take an orthonormal frame at p. Fix a simply connected nbhd U of p.
Since $R=0$, frame can be extended to a basis $\{e^i\}$ of sections of TM over U, with $\nabla e^i = 0$.
In ptc, $\nabla_{e^j} e^i - \nabla_{e^i} e^j = [e^i, e^j] = 0$.
$\implies$ can find coordinates x^i such that $e^i = \frac{\partial}{\partial x^i}$ (perhaps after shrinking U) [Frobenius]
In these coordinates $g_{ij} = \delta_{ij}$.

Computing in local coordinates

$$R_{ijk}{}^{l}\,\partial_l = R(\partial_i, \partial_j)\partial_k$$

$$= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$$

$$= \nabla_{\partial_i}(\Gamma_{jk}^{l}\partial_l) - \nabla_{\partial_j}(\Gamma_{ik}^{j}\partial_j)$$

$$= (\partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{m}\Gamma_{im}^{l} - \Gamma_{ik}^{m}\Gamma_{jm}^{l})\,\partial_l$$

[an instance of our formula $F = dA + A \wedge A$]

In normal coordinates this simplifies to

$$R_{ijk}{}^{l} = (\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik}) = \frac{1}{2}\left(\partial_i\left(g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk})\right)\right) - (i \leftrightarrow j))$$
$$= \frac{1}{2} g^{lm}(\partial_i\partial_j g_{km} + \partial_i\partial_k g_{jm} - \partial_i\partial_m g_{jk}) - (i \leftrightarrow j)$$

ie, $R_{ijkl}(x=0) = \frac{1}{2}[\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}]$ (From this, can prove the Taylor exp. (*) of g.)

How to think about R?

One approach: look for its "irreducible pieces."

Let V be an $\langle , \rangle$ space and $\underline{Riem} \subset (V^*)^4$ consist of tensors with the symmetries of R.

$O(V) \circlearrowright \underline{Riem}$ and we can ask for its irreducible pieces.

Fact: $\underline{Riem} \simeq \underset{\frac{1}{2}n(n+1)-1}{\underline{Ric^{\circ}}} \oplus \underset{\frac{1}{12}n(n+1)(n+2)(n-3)}{\underline{Weyl}} \oplus \underset{1}{\underline{S}}$ for $n \geq 3$; all irreducible for $n > 4$
($\underline{Weyl} = \underline{Weyl}^{+} \oplus \underline{Weyl}^{-}$ for $n=4$)

Ricci curvature: $Ric_{ij} = R_{kij}{}^{k}$ (symmetric)

Scalar curvature: $S = R_i{}^{i}$

Weyl curvature: $C = R - \frac{1}{n-2}\left(Ric - \frac{S}{n}g\right) \owedge g - \frac{S}{2n(n-1)}\, g \owedge g$

where $(g \owedge h)(v_1,v_2,v_3,v_4) = g(v_1,v_3)h(v_2,v_4) + g(v_2,v_4)h(v_1,v_3) - g(v_1,v_4)h(v_2,v_3) - g(v_2,v_3)h(v_1,v_4)$

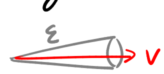
$S^2T^* \otimes S^2T^* \to S^2(\Lambda^2 T^*)$

Interpretations:

1) In normal coordinates, letting h = Euclidean metric,

$$dvol(g) = \left[1 - \frac{1}{6} Ric_{kl} x^k x^l + \cdots\right] dvol(h)$$

(pf easy beginning from $g_{ij} = \delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l$: just expand $\sqrt{\det g}$, using $\det(\mathbb{1} + \varepsilon A) = 1 + \varepsilon \operatorname{tr} A$)

This tells us about the volume of wedges of a geodesic ball. 

$\frac{vol}{vol_{Euc}} = \left[1 - \frac{\varepsilon^2}{6} Ric(v,v)\right]$

2) Integrating over the ball:

$$\int_{B_\varepsilon(0)} x^k x^l \, dvol(h) = \begin{cases} 0 & [k \neq l] \\ \frac{1}{n}\int_0^{\varepsilon} r^2 \, dvol(h) = \frac{1}{n}\int_0^{\varepsilon} r^{n+1} vol(S^{n-1})\, dr & \\ = \frac{\varepsilon^{n+2}}{n(n+2)} vol(S^{n-1}) = \frac{\varepsilon^{n+2}}{n(n+2)} vol(S^{n-1}) & [k = l] \\ = \frac{\varepsilon^2}{n+2} vol(B^n_\varepsilon) & \end{cases}$$

So, $$\frac{\mathrm{vol}(B_\varepsilon(p))}{\mathrm{vol}(B_\varepsilon(0)) \text{ in } \mathbb{R}^n} = 1 - \frac{S(p)}{6(n+2)}\varepsilon^2 + \cdots$$

Special cases: if $n=1$: $R=0$ identically

$n=2$: $R_{12}{}^{21} = \frac{1}{2}S$, other components related by symmetries

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