Def The <u>sectional curvature</u> of M <u>at p</u> is $K(p): T_pM \times T_pM \to \mathbb{R}$

$$K(X,Y) = \frac{Rm(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$$

$K(X,Y)$ only depends on the plane spanned by X,Y.

How to interpret it?

Prop Consider the 2-manifold $S_{X,Y} = \{\exp(tX+sY) : |t|<\varepsilon, |s|<\varepsilon\} \subset M$.
$K(X,Y)$ is the scalar curvature of $S_{X,Y}$ at p.

Pf A radial geodesic γ through p in M also lies in $S_{X,Y}$. Use $\tilde{\ }$ for quantities on $S_{X,Y}$.
$0 = \nabla_{\dot\gamma}\dot\gamma = \tilde\nabla_{\dot\gamma}\dot\gamma + \mathrm{II}(\dot\gamma,\dot\gamma)$ and the two terms are $\perp$ so they separately vanish.
Thus $\mathrm{II}=0$. Then $R=\tilde R$, so $K(X,Y) = \dfrac{\tilde R(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$ as desired.

Prop K determines R.

Pf Suppose $T(X,Y,Y,X)=0$ and T has the symmetries of Rm.
Then show $T=0$ [Exercise].

Def M has <u>constant curvature</u> C if $\forall p\in M, \forall X,Y\in T_pM$, $K(X,Y)=C$.

Prop $\mathbb{R}^n$ has const. curv. $C=0$.
S^n_R has const. curv. $C = 1/R^2$.
$\mathbb{H}^n_R$ has const. curv. $C = -1/R^2$.

Pf For S^n_R use $K = 1/R \cdot 1/R$
For $\mathbb{H}^n_R$ make a direct computation at a single point [Exercise], then use isometries.