Comparison Theorems

Lemma (Sturm) **If:** $u, v \in C^1([0,T]) \cap C^2((0,T))$, $u>0$ on $(0,T)$

$a : [0,T] \to \mathbb{R}$

$$\ddot{u} + au = 0 \qquad u(0) = v(0) = 0$$

$$\ddot{v} + av \geq 0 \qquad \dot{u}(0) = \dot{v}(0) > 0$$

Then: $v(t) \geq u(t)$ on $[0,T]$

Pf Take $f(t) = v(t)/u(t)$. $\lim_{t\to 0} f(t) = 1$, so need to show $\dot{f}(t) \geq 0$.

But $\dot{f}(t) = \frac{v\dot{u} - u\dot{v}}{u^2}$, and $v\dot{u}(0) - u\dot{v}(0) = 0$ while $\frac{d}{dt}(v\dot{u} - u\dot{v}) = \ddot{v}u - v\ddot{u} = (\ddot{v} + av)u \geq 0$

$\Rightarrow v\dot{u} - u\dot{v} > 0$ as needed.

Thm M Riemannian, all sectional curvatures $\leq C$

γ geodesic, $\|\dot{\gamma}\| = 1$, J normal Jacobi along γ with $J(0) = 0$

$$\Rightarrow \quad \|J(t)\| \geq \begin{cases} t\,|\nabla_t J(0)| & 0 \leq t & C = 0 \\ R\sin(t/R)\,|\nabla_t J(0)| & 0 \leq t \leq \pi R & C = 1/R^2 \\ R\sinh(t/R)\,|\nabla_t J(0)| & 0 \leq t & C = -1/R^2 \end{cases}$$

Pf Jacobi eq $\Rightarrow$
$$\frac{d^2}{dt^2}\|J(t)\| = -\frac{Rm(J,\dot{\gamma},\dot{\gamma},J)}{\|J\|} + \frac{\|\nabla_t J\|^2}{\|J\|} - \frac{\langle \nabla_t J, J\rangle^2}{\|J\|^3}$$

$$\geq -\frac{Rm(J,\dot{\gamma},\dot{\gamma},J)}{\|J\|} \qquad \text{(Cauchy-Schwarz)}$$

$$\geq -C\|J\|$$

$\left.\frac{d}{dt}\right|_{J=0} \|J(t)\| = \|\nabla_t J(0)\|$ (use $J^i = tW^i$) so can apply Sturm lemma to: $\begin{cases} u \text{ with } \ddot{u} = -Cu,\ \dot{u}(0) = 1 \\ v = \|J\| / \|\nabla_t J(0)\| \end{cases}$

This gives the result.

Cor M Riemannian, all sectional curvatures $\leq C$

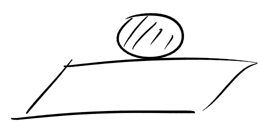
$C \leq 0 \Rightarrow M$ has no conjugate pts

$C = 1/R^2 \Rightarrow$ first conjugate pt along a geodesic occurs at distance $\geq \pi R$.

Rk Could prove more directly that $C \leq 0 \Rightarrow M$ has no conjugate pts.
Indeed, say γ geodesic with $\gamma(0)=p$, and J normal Jacobi field along γ with $J(0)=0$.
Let $f(t) = \|J(t)\|^2$; then
$$f''(t) = 2\langle \nabla_t\nabla_t J, J\rangle + \|\nabla_t J\|^2$$
$$= -2Rm(J,\dot\gamma,\dot\gamma,J) + \|\nabla_t J\|^2 \geq 0$$
and $f(0) = f'(0) = 0$, but $\nabla_t J(0) \neq 0$ (use normal coords), so $f''(0) > 0$.
Then $f(t) > 0 \ \forall t > 0$.

Lemma $\tilde M, M$ Riem, $\tilde M$ complete, $\pi : \tilde M \to M$ local isometry $\Rightarrow M$ complete, π is covering map

Pf In steps:



1) Path lifting for geodesics: if $p \in M$, $\tilde p \in \pi^{-1}(p)$, γ geodesic $\gamma(0)=p$

$$\begin{tikzcd} & \tilde M \arrow[d, "\pi"] \\ {[0,T]} \arrow[ur, "\tilde\gamma"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

then γ has unique lift given by: $\tilde\gamma(t) = \exp\left(t\, \pi_*^{-1}(\dot\gamma(0))\right)$

(local isometry property shows $\pi \circ \tilde\gamma = \gamma$)

$\Rightarrow$ can extend γ to $\pi \circ \tilde\gamma$, defined on $[0,\infty]$ since $\tilde M$ complete; hence M complete

2) Surjectivity: given p in image of π, and $q \in M$, connect p to q by a geodesic segment; lifting this segment to $\tilde M$ shows that q is also in image of π

3) Even covering: take a geodesic ε-ball U around p.
Take metric ε-balls $\tilde U_\alpha$ around preimages $\tilde p_\alpha \in \pi^{-1}(p)$.
A geodesic $\tilde\gamma$ from $\tilde p_\alpha$ to $\tilde p_\beta$ has the same length as $\gamma = \pi \circ \tilde\gamma$, which must be $\geq 2\varepsilon$ since γ exits and re-enters U. So $d(\tilde p_\alpha, \tilde p_\beta) \geq 2\varepsilon$, hence $\tilde U_\alpha \cap \tilde U_\beta = \emptyset$.

To show $\pi^{-1}(U) = \bigcup_\alpha \tilde U_\alpha$, hard direction is $\pi^{-1}(U) \subset \bigcup_\alpha \tilde U_\alpha$; use lift of radial geodesics. To show π is diffeo on $\tilde{U}_\alpha$, construct its inverse using lifts of radial geodesics.

__Thm__ (Cartan-Hadamard) M complete, all sectional curv. ≤ 0

$\Longrightarrow$ universal cover of M is diffeomorphic to $\mathbb{R}^n$.

(So if M also simply conn. then M is diffeo to $\mathbb{R}^n$.)

__Pf__ $\exp: (T_pM, \exp^* g) \longrightarrow (M, g)$ is local isometry.

$(T_pM, \exp^* g)$ is geodesically complete at 0 hence complete

So, the last lemma shows $\exp: T_pM \rightarrow M$ is a <u>covering map</u>!