Weitzenbock formula

Def in ON-basis $\{e_i\}$, $a_j^*(\omega) = e_j \wedge \omega$, $a_i(\omega) = \iota_{e_i}\omega$

Lemma $d = a_i^* \nabla_{e_i}$ $\quad d^* = -a_i \nabla_{e_i}$

Pf Fix normal coords and say $e_i = \partial_i$ at p. Then just compute!

$$\left[\text{e.g.}\quad d(f \cdot e_1 \wedge e_2) = \partial_3 f \cdot e_3 \wedge e_1 \wedge e_2\right]$$

$$\left[\begin{aligned}\text{e.g.}\quad d^*(f\cdot e_1\wedge e_2) &= \star d \star (f \cdot e_1 \wedge e_2) = \star d(f\cdot e_3) = \star(\partial_1 f\, e_1\wedge e_3 + \partial_2 f\, e_2 \wedge e_3)\\ &= -\partial_1 f \cdot e_2 + \partial_2 f \cdot e_1\end{aligned}\right]$$

Def Curvature endomorphism: $\hat{R} : \Omega^p(M) \to \Omega^p(M)$ $\quad \hat{R} = -R_{ijkl}\, a_i^* a_j\, a_k^* a_l$

Def/Prop $\nabla^* =$ formal adjoint of ∇ $\quad \Omega^p(M) \underset{\nabla}{\overset{\nabla^*}{\leftrightarrows}} \mathcal{E}(\Lambda^p T^* \otimes T^*)$

determined by $\langle \alpha, \nabla^* \beta\rangle = \langle \nabla\alpha, \beta\rangle - d^*\langle \alpha, \beta\rangle$ $\quad \alpha \in \Omega^p(M)$, $\beta \in \mathcal{E}(\Lambda^p T^* \otimes T^*)$

Lemma M compact $\Longrightarrow$ $\langle \alpha, \nabla^*\beta\rangle_{L^2} = \langle \nabla\alpha, \beta\rangle_{L^2}$.

Pf $\displaystyle\int_M (\langle \alpha, \nabla^*\beta\rangle - \langle\nabla\alpha,\beta\rangle)\,\mathrm{vol} = -\int_M d^*\langle\alpha,\beta\rangle\,\mathrm{vol} = -\int_M \star d \star \langle \alpha,\beta\rangle \cdot \mathrm{vol} = \pm\int_M d\star\langle\alpha,\beta\rangle = 0$

Thm $\Delta = \nabla^*\nabla + \hat{R}$.

Pf Fix normal coords and say $e_i = \partial_i$ at p.

Then $d^*d = -a_i \nabla_{e_i}(a_j^* \nabla_{e_j}) = -a_i a_j^* \nabla_{e_i}\nabla_{e_j} = -\nabla_{e_k}\nabla_{e_k} + a_j^* a_i \nabla_{e_i}\nabla_{e_j}$

$dd^* = -a_j^* \nabla_{e_j}(a_i \nabla_{e_i}) = -a_j^* a_i \nabla_{e_j}\nabla_{e_i}$

so $\Delta = \underbrace{-\nabla_{e_k}\nabla_{e_k}}_{} + \underbrace{a_j^* a_i(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})}_{}$

$\hookrightarrow\; = a_j^* a_i\,(R_{ijkl}\, a_k^* a_l)$ since $[\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}]e^l = R_{ijkl}\, e^k$

$= \nabla^*\nabla$ since

$$\begin{aligned}\langle \nabla^*\nabla\alpha, \beta\rangle &= \langle\nabla\alpha,\nabla\beta\rangle - d^*\langle\nabla\alpha,\beta\rangle\\ &= \langle \nabla_{e_k}\alpha, \nabla_{e_k}\beta\rangle - \partial_k\langle\nabla_{e_k}\alpha,\beta\rangle\\ &= -\langle\nabla_{e_k}\nabla_{e_k}\alpha,\beta\rangle\end{aligned}$$