Lecture 6

12 Sep 2012

Housekeeping:

My office hours: today (special) 5–6p (RLM 9.134)

Monday 11a–12 noon

Thursday 2–3p

Reminder: table of indef. $\int$ in text p. 398 (?)

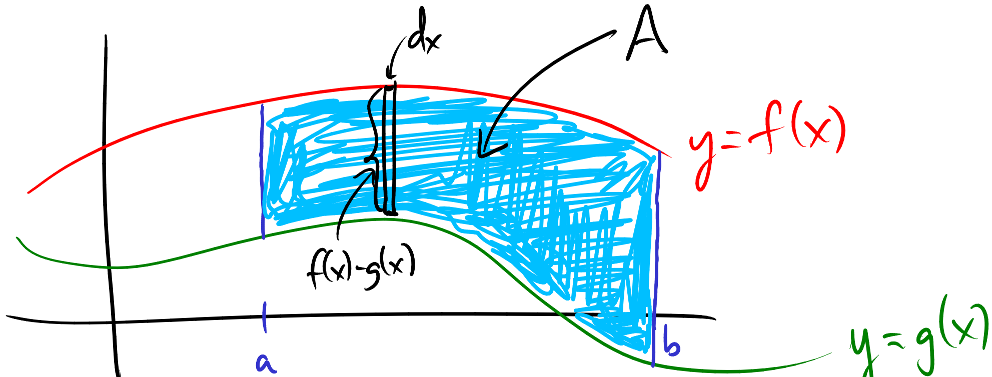
e.g. $$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

Areas between curves (Ch 6.1)

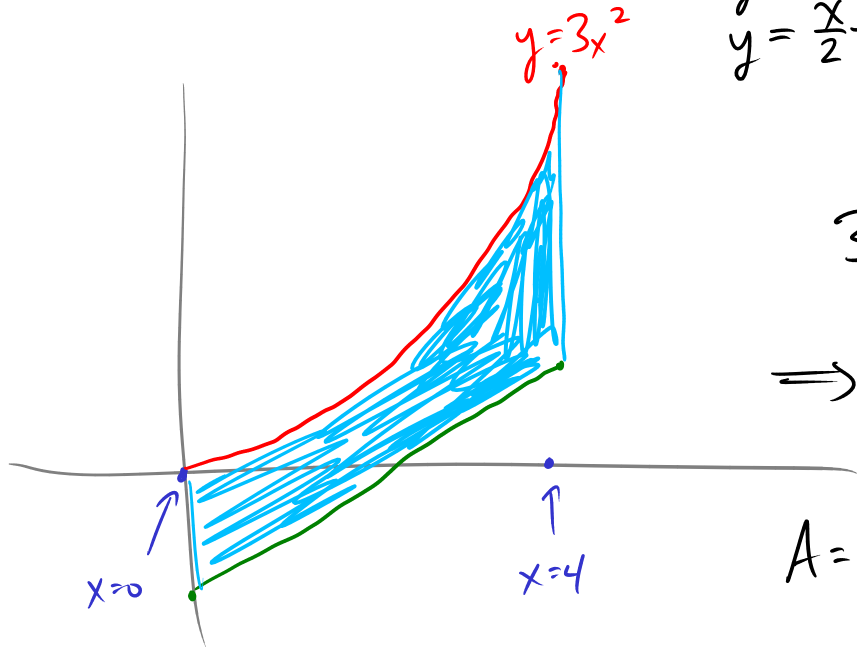
Two curves $y = f(x)$ and $y = g(x)$.

Suppose $f(x) > g(x)$ for $x \in [a,b]$. $(a \leq x \leq b)$.



The area A is given by $\int_a^b [f(x) - g(x)]\, dx$.

Ex Find the area between the curves $y = 3x^2$, $y = \frac{x}{2} - 1$ from $x=0$ to $x=4$.



$3x^2 > \frac{x}{2} - 1$ for $x \in [0,4]$

$\implies$ the area is

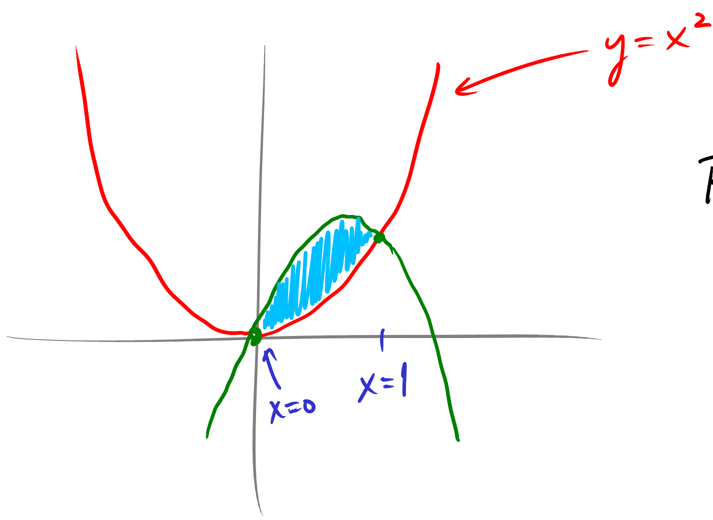
$$A = \int_0^4 3x^2 - \left(\frac{x}{2} - 1\right)\, dx$$

$$= \int_0^4 3x^2 - \frac{x}{2} + 1 \, dx$$

$$= x^3 - \frac{1}{4}x^2 + x \Big]_0^4$$

$$= (64 - 4 + 4) - (0) = \underline{\underline{64}}$$

Ex Find the area of the region between $y = x^2$ and $y = 2x - x^2$.



Find the points of intersection:

$$x^2 = 2x - x^2$$

$$0 = 2x - 2x^2$$

$$0 = 2(x - x^2)$$

$$0 = 2 \cdot x(1-x)$$

$x=0$ or $x=1$

plug back in: intersection pts are $(0,0)$ and $(1,1)$

$$A = \int_0^1 (2x - x^2) - (x^2)\, dx$$

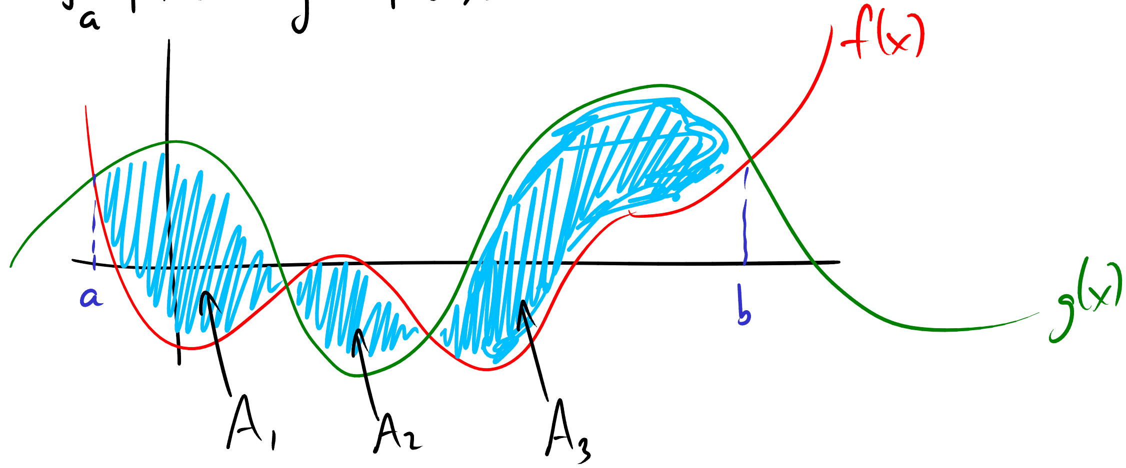
$$= \int_0^1 2x - 2x^2 \, dx$$

$$= x^2 - \tfrac{2}{3}x^3 \Big]_0^1$$

$$= \frac{1}{3}$$

A rule that finds the area between $y=f(x)$ and $y=g(x)$ no matter which is bigger:

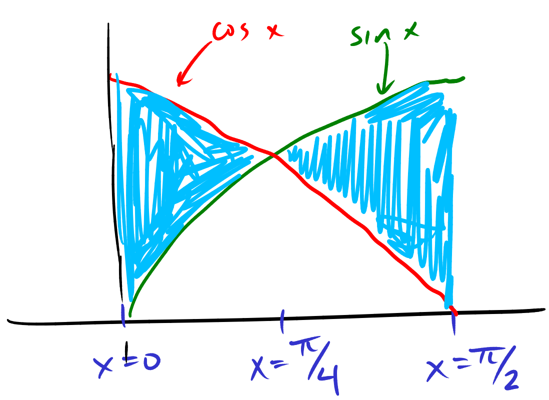
$$A = \int_a^b |f(x) - g(x)| \, dx$$



$$A = A_1 + A_2 + A_3$$

To actually calculate this, usually need to break into pieces.

Ex Find the area of the region between $y = \sin x$ and $y = \cos x$ where x ranges between $x = 0$ and $x = \pi/2$.



$$A = \int_0^{\pi/2} |\sin x - \cos x| \, dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx$$

$$= \sin x + \cos x \Big]_0^{\pi/4} + (-\sin x - \cos x) \Big]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) - (0+1)\right] + \left[(-1-0) - \left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)\right]$$

$$= 4 \cdot \frac{\sqrt{2}}{2} - 2$$

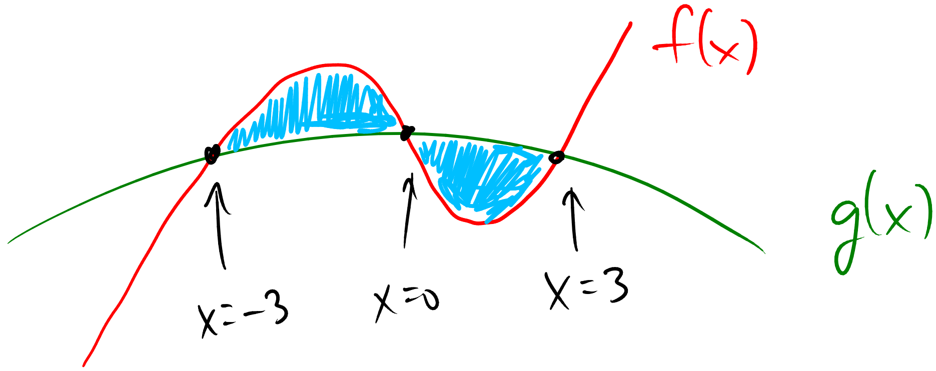
$$= \underline{\underline{2\sqrt{2} - 2}}$$

Ex Find the area of the region between the curves:

$$y = x^3 - x^2 - 7x - 4 = f(x)$$
$$y = -x^2 + 2x - 4 = g(x)$$

Points of intersection:

$$x^3 - x^2 - 7x - 4 = -x^2 + 2x - 4$$
$$x^3 - 9x = 0$$
$$x(x^2 - 9) = 0$$
$$x(x+3)(x-3) = 0 \implies \text{int. are } \underline{x = 0, 3, -3}$$



Area is $\int_{-3}^{3} |f(x) - g(x)|\, dx$

$$= \int_{-3}^{0} f(x) - g(x)\, dx + \int_{0}^{3} g(x) - f(x)\, dx$$

$$= \int_{-3}^{0} x^3 - 9x\, dx + \int_{0}^{3} 9x - x^3\, dx$$

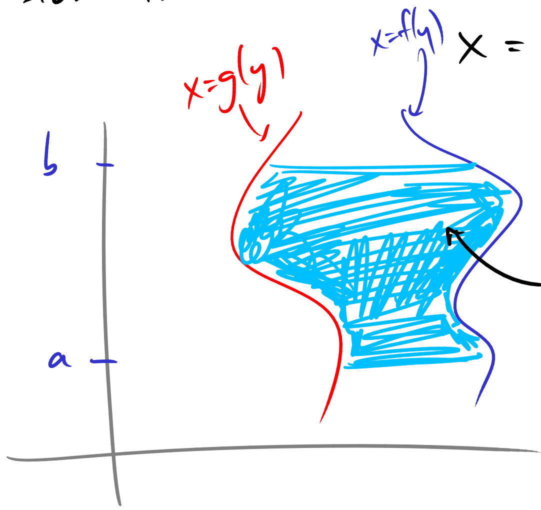
$$= \left(\frac{1}{4}x^4 - \frac{9}{2}x^2\Big]_{-3}^{0}\right) + \left(-\frac{1}{4}x^4 + \frac{9}{2}x^2\Big]_{0}^{3}\right)$$

$$= -\left(\frac{1}{4}81 - \frac{81}{2}\right) + \left(-\frac{81}{4} + \frac{81}{2}\right)$$

$$= 81\left(-\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2}\right)$$

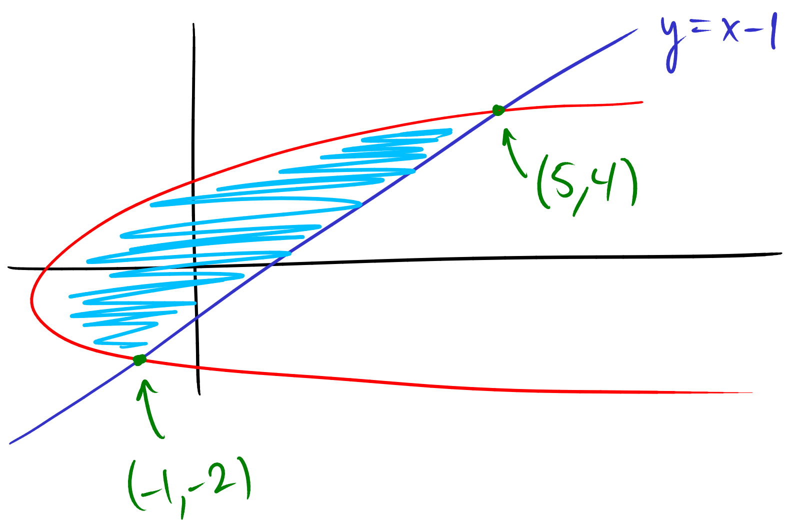
$$= \underline{\underline{\frac{81}{2}}}$$

Can also consider two curves: $x = f(y)$, $x = g(y)$



$$A = \int_a^b |f(y) - g(y)|\, dy$$

<u>Ex</u> Find the area between the parabola $y^2 = 2x + 6$ and $y = x - 1$



Write the curves in form $x = f(y)$, $x = g(y)$: $x = \frac{1}{2}y^2 - 3$, $x = y + 1$

$$A = \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) \, dy = \cdots = \underline{\underline{18}}$$