Lecture 12 | 28 Sep 2012

Exam Tue Oct 9 7pm

Covers material thru next Fri's lecture, i.e. thru+including HW07

18 problems

Last time: Partial fractions

$$\int \frac{P(x)}{Q(x)}\,dx \qquad P, Q \text{ poly.} \qquad \text{e.g.} \quad \int \frac{x^4+4x^3+2x-17}{x^3+2x^2+x}\,dx$$

First, do polynomial long division (if degree(P) $\geq$ degree(Q))

Then factor the denominator and use that to split up the fraction.

Can even use this on simple-looking things:

$$\text{e.g.} \quad \int \frac{1}{x^2+x}\,dx \qquad\qquad x^2+x = x(x+1)$$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Mult. by $x(x+1)$ both sides:

$$1 = A(x+1) + Bx$$

Plug in $x=-1 \rightsquigarrow 1 = -B$

$x = 0 \rightsquigarrow 1 = A$

$\longrightarrow$ $A=1$, $B=-1$

So $$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x(x+1)}\,dx = \ln|x| - \ln|x+1| + C = \underline{\underline{\ln\left|\frac{x}{x+1}\right| + C}}$$

also applies e.g. to $\displaystyle\int \frac{dx}{x^2-4}$ ...

Strategy For Integration (Ch 7.5)

1) Basic integration formulas

$$\int x^n\,dx = \dots$$
$$\int \sin x\,dx = \dots$$
$$\int \frac{1}{x}\,dx = \dots$$
$$\int \frac{1}{1+x^2}\,dx = \dots$$

(table in Ch 7.5)

2) Simplify the integrand (using either algebra or trig identities)

$$\int \sqrt{x}(1+\sqrt{x})\,dx = \int \sqrt{x} + x\,dx = \dots$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta\,d\theta = \int \sin\theta\cos\theta\,d\theta = \frac{1}{2}\int \sin 2\theta\,d\theta$$
$$= \dots$$

3) "Easy" substitutions

$$\int \frac{x}{x^2-1}\,dx \qquad \begin{aligned} u &= x^2-1 \\ du &= 2x\,dx \end{aligned} \quad \dots$$

4) Classify the integrand:

a) Trig $\left[\sin^a x \cos^b x,\ \tan^a x \sec^b x,\ \cot^a x \csc^b x\right]$

– use rules from last week

b) Rational function $\left(\frac{P(x)}{Q(x)}\right)$ – partial fractions

c) Product of 2 different "kinds" of function $\qquad \int x e^x\, dx$

– int. by parts

d) Radicals – $\sqrt{\pm x^2 \pm a^2}$ try trig sub

$\sqrt[n]{ax+b}$ try $u = \sqrt[n]{ax+b}$ sub

5) Try again:

a) look for a clever substitution

b) $\int$ by parts – even if you have to take u = the whole integrand, $dv = dx$

(e.g. $\int \tan^{-1} x\, dx$ can be done this way)

c) algebraic manipulations (e.g. $\int \frac{dx}{1-\cos x}$: mult. top, bottom by $1+\cos x$)

d) try to relate it to one you've done before...

e) combine several methods..

Ex $\displaystyle \int \frac{1}{9+x^2}\, dx = \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right)$

$\left[\text{or: use } x = 3u \text{ and } \int \frac{du}{1+u^2} = \tan^{-1}(u)\right]$

Ex $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \tan^3 x \sec^3 x \, dx$

odd # of powers of $\tan x$ $\Rightarrow$ $\displaystyle\int \tan^2 x \sec^2 x \,(\sec x \tan x \, dx)$

$u = \sec x$

$\longrightarrow \displaystyle\int (u^2 - 1)\, u^2 \quad du$

Ex $\displaystyle\int e^{\sqrt{x}}\, dx$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$= \displaystyle\int e^u \cdot 2u \, du$

then $\int$ by parts...

Ex $\displaystyle\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\, dx$ partial fractions

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Ex $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$

$u = \ln x$

$du = \dfrac{dx}{x}$

$\displaystyle\int \frac{du}{\sqrt{u}} = \int du \; u^{-1/2} = \cdots$

Ex $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\, dx$ mult top, bottom by $\sqrt{1-x}$

$= \displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\, dx = \int \frac{1}{\sqrt{1-x^2}}\, dx - \int \frac{x}{\sqrt{1-x^2}}\, dx$ ← u-sub $u = 1 - x^2$

$$= \sin^{-1} x \quad - \quad \cdots$$

Ex $\int (x + \sin x)^2 \, dx$

$$= \int x^2 + 2x \sin x + \sin^2 x \, dx$$

easy — int. by parts — half-angle identities

Ex

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{dx}{\left(\sqrt{1-x^2}\right)^3}$$

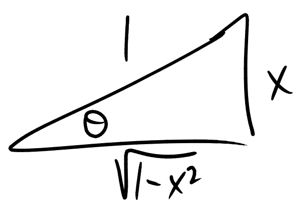
$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$= \int \frac{\cos\theta \, d\theta}{\cos^3\theta} = \int \sec^2\theta \, d\theta$$

$$= \tan\theta + C$$

re-express in terms of x:



$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

Ex

$$\int \frac{1}{\sqrt{4y^2 - 4y - 3}} \, dy = \int (4y^2 - 4y - 3)^{-1/2} \, dy$$

<u>Complete the square!</u>

$$4y^2 - 4y - 3 = (Ay + B)^2 + \text{const.}$$

$$= A^2 y^2 + 2ABy + B^2 + \text{const.}$$

$4 = A^2 \qquad -4 = 2AB \qquad A = 2$

$B = -1$

$$4y^2 - 4y - 3 = (2y-1)^2 - 4$$

so $\int = \int \frac{dy}{\sqrt{(2y-1)^2 - 4}}$ $\qquad u = 2y - 1$

$dy = \frac{1}{2} du$

$$= \int \frac{\frac{1}{2} du}{\sqrt{u^2 - 4}}$$

$u = 2 \sec \theta$

$$= \cdots$$

$$= \frac{1}{2} \log\left(1 - 2y - \sqrt{4y^2 - 4y - 3}\right)$$