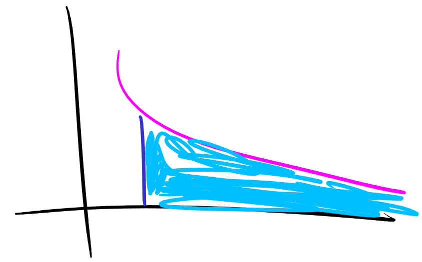
Lecture 14

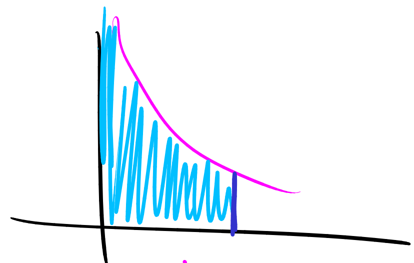
3 Oct 2012

Last time: <u>improper integrals</u>

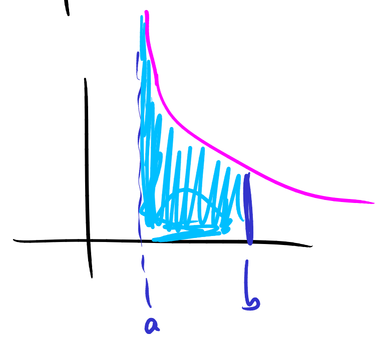
$$\int_a^\infty \frac{1}{x^p}\,dx \quad (a>0) \quad \begin{cases} \underline{\text{converges}} \text{ if } p>1 \\ \underline{\text{diverges}} \text{ if } p\le 1 \end{cases}$$



$$\int_0^a \frac{1}{x^p}\,dx \quad (a>0) \quad \begin{cases} \underline{\text{converges}} \text{ if } p<1 \\ \underline{\text{diverges}} \text{ if } p\ge 1 \end{cases}$$



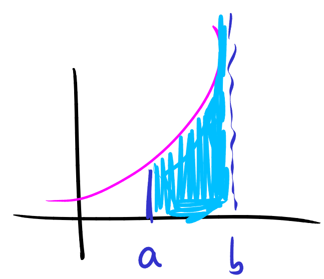
Similarly, $\displaystyle\int_a^b \frac{1}{(x-a)^p}\,dx \quad \begin{cases} \underline{\text{conv}} & p<1 \\ \underline{\text{div}} & p\ge 1 \end{cases}$



<u>Ex</u> $\displaystyle\int_{-4}^{-3} \frac{1}{(x+4)^2}\,dx$ diverges

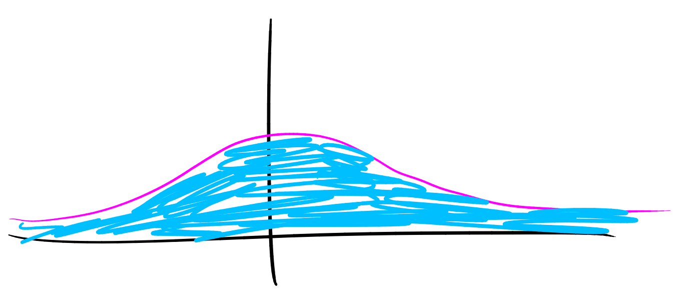
(to <u>check</u> this, just calculate $\displaystyle\lim_{a\to -4^+} \int_a^{-3} \frac{1}{(x+4)^2}\,dx$)

Similarly, $\displaystyle\int_a^b \frac{1}{(x-b)^p}\,dx \quad (p\ge 0) \quad \begin{cases} \underline{\text{conv}} & p<1 \\ \underline{\text{div}} & p\ge 1 \end{cases}$



One more kind of improper $\int$:

$$\int_{-\infty}^{\infty} f(x)\,dx$$



This is defined by splitting it up:

$$\int_{-\infty}^{a} = \lim_{t\to-\infty} \int_{t}^{a}$$

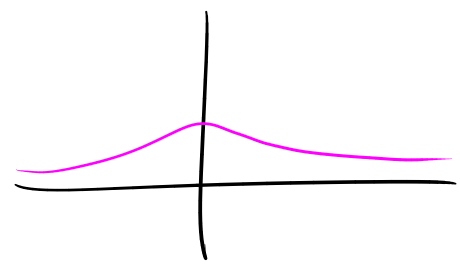
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \left(\lim_{t\to-\infty} \int_{t}^{0} f(x)\,dx\right) + \left(\lim_{t\to\infty} \int_{0}^{t} f(x)\,dx\right)$$

If <u>both</u> of these limits exist, $\neq \pm\infty$, we say the $\int$ is <u>convergent</u>

Otherwise it's <u>divergent</u>

<u>Ex</u> $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$



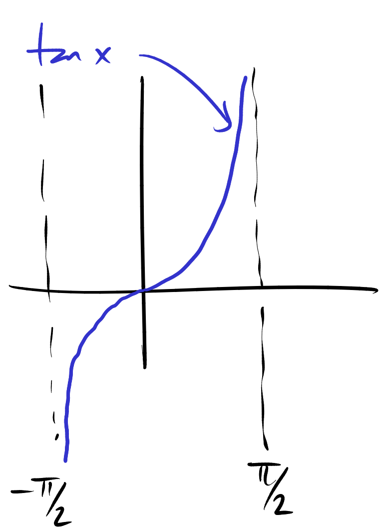
$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

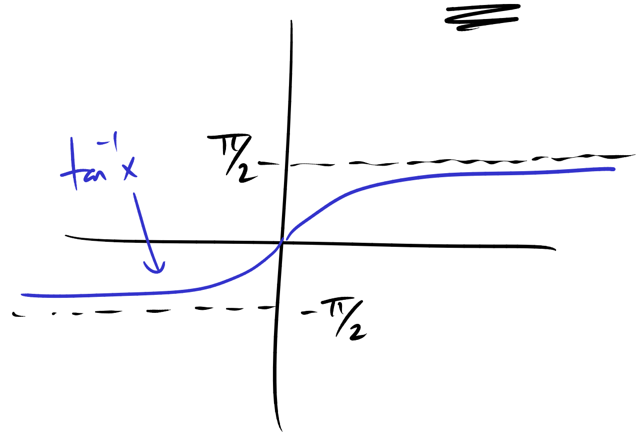
$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \tan^{-1}x \Big]_{t}^{0} + \lim_{t\to\infty} \tan^{-1}x \Big]_{0}^{t}$$

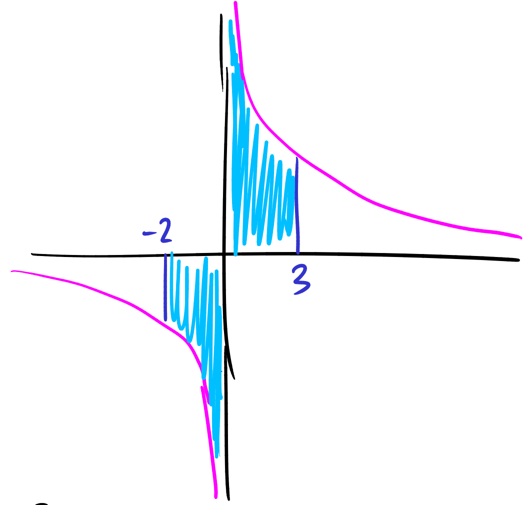
$$= \lim_{t\to-\infty} \left(-\tan^{-1}t\right) + \lim_{t\to\infty} \left(\tan^{-1}t\right) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \underline{\underline{\pi}}$$





Ex $\displaystyle\int_{-2}^{3} \frac{1}{x}\,dx$



Improper b/c $\frac{1}{x}$ has vert. asymp at $x=0$

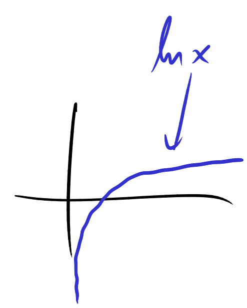
$$\int_{-2}^{3} \frac{1}{x}\,dx = \int_{-2}^{0} \frac{1}{x}\,dx + \int_{0}^{3} \frac{1}{x}\,dx$$

$$= \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x}\,dx + \lim_{t\to 0^+} \int_{t}^{3} \frac{1}{x}\,dx$$

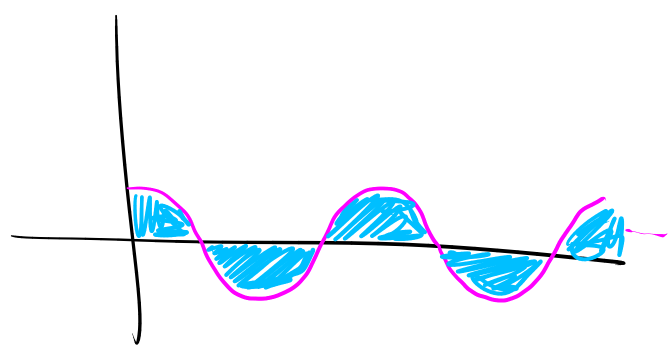
$$= \lim_{t\to 0^-} \ln|x| \Big|_{-2}^{t} + \lim_{t\to 0^+} \ln|x| \Big|_{t}^{3}$$

$$= \Big(\lim_{t\to 0^-} \underbrace{\ln|t|}_{-\infty} - \ln 2\Big) + \lim_{t\to 0^+} \big(\ln 3 - \underbrace{\ln|t|}_{-\infty}\big)$$

the 2 limits don't exist $\Rightarrow$ divergent



Ex $\displaystyle\int_{0}^{\infty} \cos x\,dx$



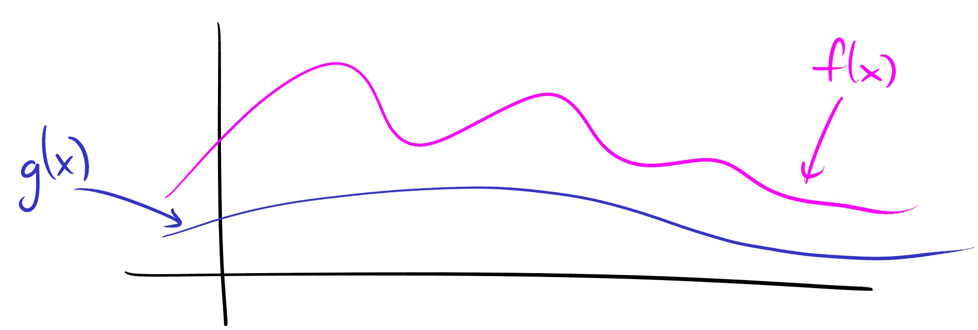
$$= \lim_{t\to\infty} \int_{0}^{t} \cos x\,dx$$

$$= \lim_{t\to\infty} \sin t\,\Big]_{0}^{t}$$

$$= \lim_{t\to\infty} \sin t$$

does not exist — so $\int_{0}^{\infty} \cos x\,dx$ is divergent

Comparison Theorem



Suppose $0 \le g(x) \le f(x)$ for all $x \ge a$

and $\int_a^\infty f(x)\,dx$ converges

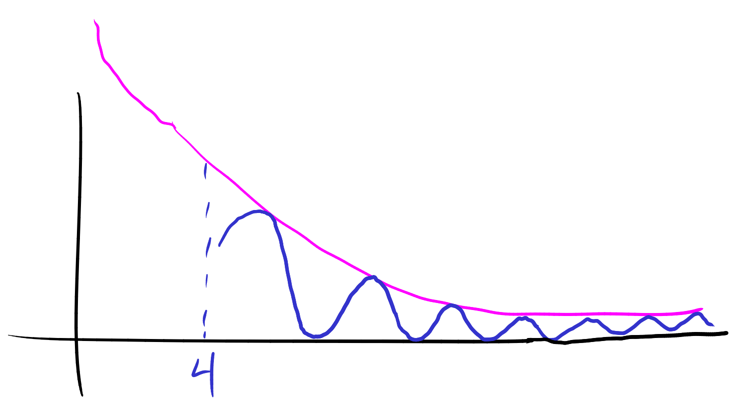
then $\int_a^\infty g(x)\,dx$ also converges.

Ex Does $\displaystyle\int_4^\infty \frac{\sin^2(x)}{x^7}\,dx$ converge?

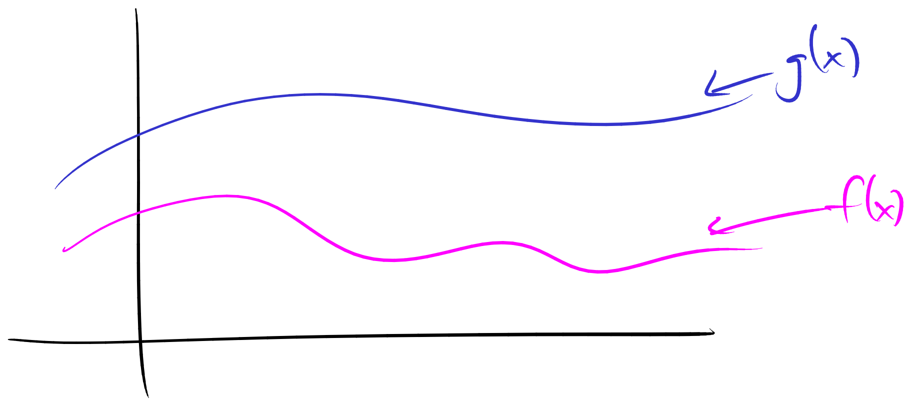
We know that $\displaystyle\int_4^\infty \frac{1}{x^7}\,dx$ converges $(7>1)$

And $0 \le \dfrac{\sin^2(x)}{x^7} \le \dfrac{1}{x^7}$ for all $x \ge 4$

So, by comparison theorem, $\displaystyle\int_4^\infty \frac{\sin^2(x)}{x^7}\,dx$ converges.



Also:



If $0 \leq f(x) \leq g(x)$

and $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges