Lecture 15 | 5 Oct 2012

Last time: improper integrals

So far, we always looked at functions $f(x)$
only one variable

In reality, things often dep. on >1 variable —

e.g. the heat index depends on temperature and humidity
$f(x,y)$ — x — y

price of cheese depends on supply and demand

productivity depends on supply of labor and supply of capital

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Partial derivatives (Ch —)

If we have $f(x,y)$ we ask:

- How does $f(x,y)$ change if we vary x and hold y fixed?

Define $$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$ "partial deriv. of f with respect to x"

To calculate it, just treat y as a constant and differentiate as usual with resp. to x.

Ex If $f(x,y) = x^2\sin(y)$ then $\frac{\partial f}{\partial x} = \underline{2x\sin(y)}$

Ex If $f(x,y) = x^2\sin(y)$ then $\frac{\partial f}{\partial x}$ at $(x,y)=(1,\pi/2)$ is $2\cdot 1\cdot \sin(\frac{\pi}{2}) = \underline{2}$

Similarly, $$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Ex $f(x,y) = 4x^2y + 7\sin(x)$

$$\frac{\partial f}{\partial x} = 8xy + 7\cos x \qquad \frac{\partial f}{\partial y} = 4x^2$$

Can also look at 2nd deriv:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 8y - 7\sin x \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 8x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 8x \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Note: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$! This always happens (whenever both are continuous)

Ex $f(x,y) = \sin(xy)$

$$\frac{\partial f}{\partial x} = y\cos(xy) \qquad \frac{\partial f}{\partial y} = x\cos(xy)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = y \cdot \frac{\partial}{\partial x}\left(\cos(xy)\right) = y \cdot \left(-y\sin(xy)\right) = -y^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = x\frac{\partial}{\partial y}\left(\cos(xy)\right) = x \cdot \left(-x\sin(xy)\right) = -x^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\cos(xy)\right) = \cos(xy) + x\frac{\partial}{\partial x}\left(\cos(xy)\right)$$

$$= \cos(xy) + x\left(-y\sin(xy)\right)$$

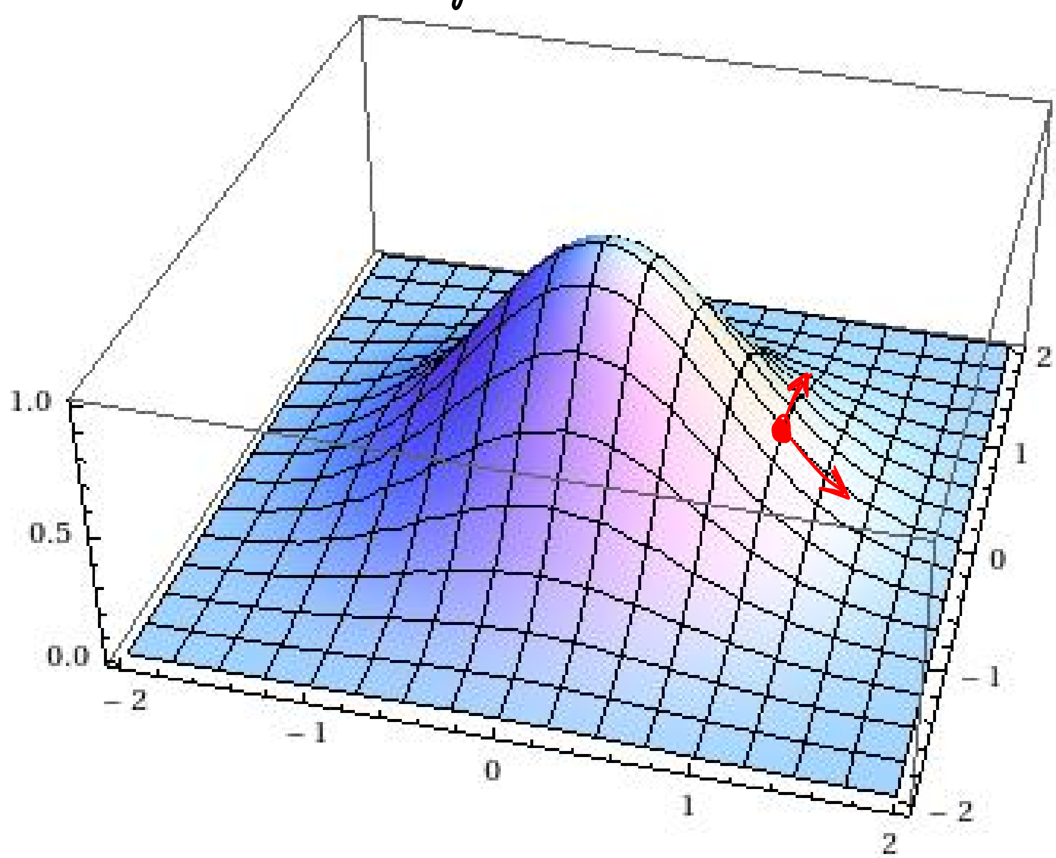
$$= \cos(xy) - xy\sin(xy)$$

Picturing the partial derivative:

Say $f(x,y) = e^{-x^2-y^2}$ $\qquad \frac{\partial f}{\partial x} = -2xe^{-x^2-y^2}$ $\qquad \frac{\partial f}{\partial y} = -2ye^{-x^2-y^2}$

So, at $(x,y) = (1,0)$: $\quad \frac{\partial f}{\partial x} = -2(1)e^{-1} = \underline{\underline{\frac{-2}{e}}} < 0$

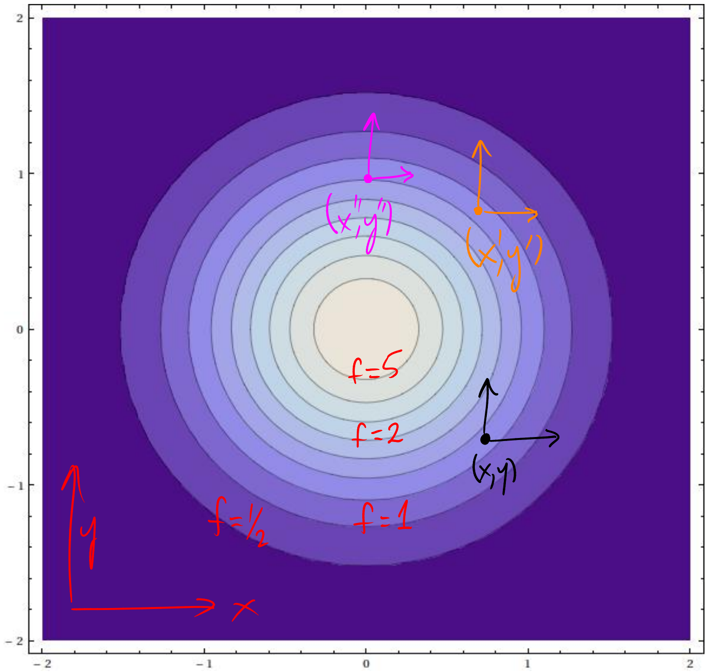
$$\frac{\partial f}{\partial y} = -2(0)e^{-1} = \underline{\underline{0}}$$



If we increase x a little bit we go downhill $\Rightarrow \frac{\partial f}{\partial x} < 0$

If we increase y a little bit we go neither up nor down $\Rightarrow \frac{\partial f}{\partial y} = 0$

Contour plot:



Contour lines are loci where $f(x,y)$ takes some constant value

At the marked point (x,y): $\frac{\partial f}{\partial x} < 0$ $\quad \frac{\partial f}{\partial y} > 0$

At (x',y'): $\frac{\partial f}{\partial x} < 0$ $\quad \frac{\partial f}{\partial y} < 0$

At (x'',y''): $\frac{\partial f}{\partial x} = 0$ $\quad \frac{\partial f}{\partial y} < 0$

(because moving along a line of constant y moves tangent to the contour-line)

Ex Suppose $z(x,y)$ is defined by

$$x^2+y^2+z^2x=0$$

What is $\frac{\partial z}{\partial x}$?

(So x and y are the variables while z is a function of x and y)

Implicit differentiation — apply $\frac{\partial}{\partial x}$ to the whole eq:

$$\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial x}(y^2)+\frac{\partial}{\partial x}(z^2x)=0$$

$$2x + 0 + z^2 + 2zx\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{2x + z^2}{2zx}$$