Lecture 17

10 Oct 2012

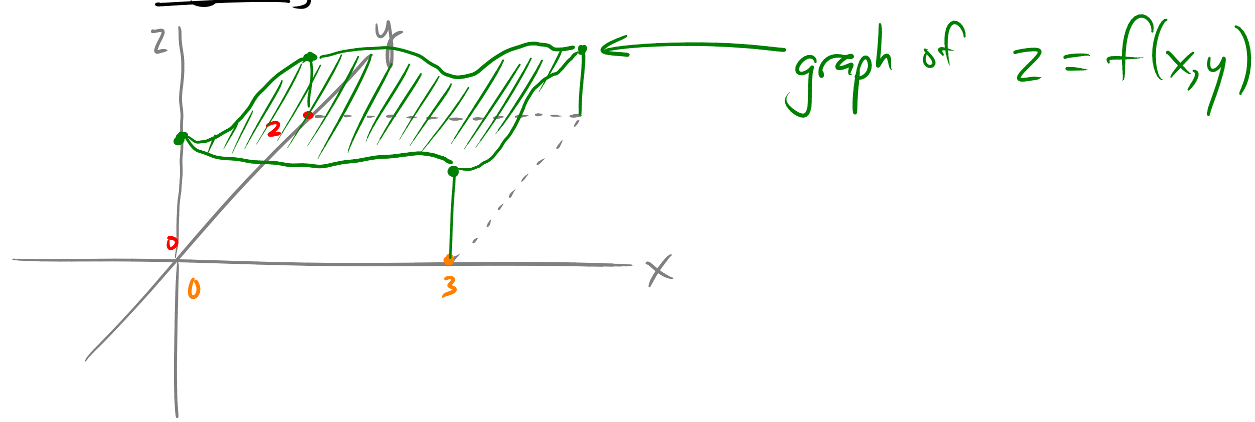
Last time: partial derivatives

$$f(x,y) \rightsquigarrow \frac{\partial f}{\partial x} = f_x \rightsquigarrow \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

$$f(x,y) \rightsquigarrow \frac{\partial f}{\partial y} = f_y \rightsquigarrow \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

these two are equal!

How about integrating a function of 2 variables?



Q: What is the total volume under this graph

$$\left(\begin{array}{ll} \text{i.e. the volume between the graph of} & z = f(x,y) \\ \text{and the xy-plane} & z = 0 \end{array} \right) ?$$

Cut by planes at fixed y: $\quad V = \int_0^2 A(y)\, dy$ cross section area

$A(y) = \int_0^3 f(x,y)\, dx$ gives the cross sec. area

So $\quad V = \int_0^2 \left[\int_0^3 f(x,y)\, dx \right] dy$

Ex Suppose $f(x,y) = 4xy + 3x^2$

Then $$V = \int_0^2 \left[\int_0^3 (4xy + 3x^2)\, dx \right] dy$$

$$= \int_0^2 \left(2x^2 y + x^3 \Big|_{x=0}^{x=3} \right) dy$$

$$= \int_0^2 (18y + 27)\, dy$$

$$= 9y^2 + 27y \Big]_{y=0}^{y=2}$$

$$= 36 + 54 = \underline{\underline{90}}$$

We could also try doing the $\int$'s in the <u>other order</u>:

$$V = \int_0^3 \left[\int_0^2 4xy + 3x^2\, dy \right] dx$$

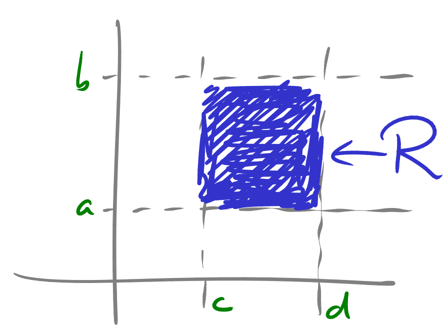
$$= \int_0^3 \left[2xy^2 + 3x^2 y \Big|_{y=0}^{y=2} \right) dx$$

$$= \int_0^3 8x + 6x^2\, dx$$

$$= 4x^2 + 2x^3 \Big|_0^3 = 36 + 54 = \underline{\underline{90}}$$

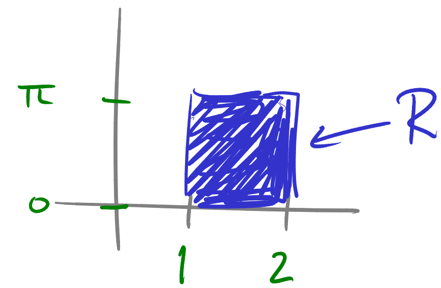
This gives the <u>same</u> answer — can choose either order ("Fubini's Theorem")

$$\int_a^b \left[\int_c^d f(x,y)\, dx \right] dy = \int_c^d \left[\int_a^b f(x,y)\, dy \right] dx$$



We also write this as $\displaystyle\iint_R f(x,y)\, dA$

Ex If $R = \{1 \le x \le 2,\ 0 \le y \le \pi\}$
and $f(x,y) = y\sin(xy)$
what is $\iint_R f(x,y)\, dA$?



It is $\int_0^{\pi} \left[\int_1^2 y\sin(xy)\, dx \right] dy$

(Or: $\int_1^2 \left[\int_0^{\pi} y\sin(xy)\, dy \right] dx$, but that's harder to calculate)

Now,
$$\int_1^2 y\sin(xy)\, dx = y \cdot \int_1^2 \sin(xy)\, dx$$
$$= y \cdot \left(-\frac{1}{y}\cos(xy) \right) \Big]_{x=1}^{x=2}$$
$$= -\cos(xy) \Big]_{x=1}^{x=2}$$
$$= -\cos(2y) + \cos(y)$$

So,
$$\int_0^{\pi} \left[\int_1^2 y\sin(xy)\, dx \right] dy$$
$$= \int_0^{\pi} \left(-\cos(2y) + \cos(y) \right) dy$$
$$= \left(-\frac{1}{2}\sin(2y) + \sin(y) \right) \Big]_{y=0}^{y=\pi}$$
$$= \underline{\underline{0}}$$

Ex Find the volume of the solid which lies under the graph of

$$z = f(x,y) = 4 + x^2 - y^2$$

and over the rectangle $\left.\begin{array}{c} -1 \le x \le 1 \\ 0 \le y \le 2 \end{array}\right]$ (call this rectangle R)

$$V = \iint_R f(x,y)\, dA$$

$$= \iint_R 4 + x^2 - y^2 \, dA$$

$$= \int_{-1}^{1} \left[\int_0^2 4 + x^2 - y^2 \, dy \right] dx$$

$$= \int_{-1}^{1} \left(4y + x^2 y - \tfrac{1}{3} y^3 \Big|_{y=0}^{y=2} \right) dx$$

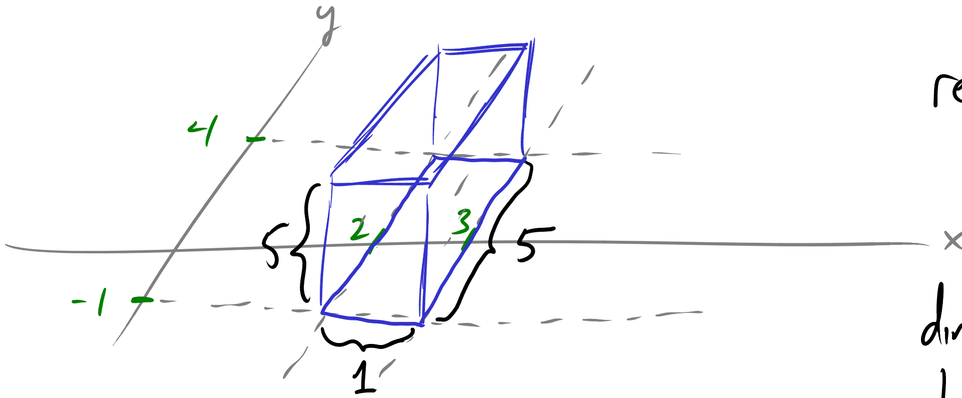
$$= \int_{-1}^{1} 8 + 2x^2 - \tfrac{8}{3} \, dx$$

$$= \int_{-1}^{1} \tfrac{16}{3} + 2x^2 \, dx$$

$$= \tfrac{16}{3} x + \tfrac{2}{3} x^3 \Big|_{-1}^{1}$$

$$= 6 + 6 = \underline{\underline{12}}$$

Ex Find $\int_{-1}^{4} \int_{2}^{3} 5 \, dx \, dy$ by interpreting it as a volume.



rectangular prism
(box)

dimensions $1 \times 5 \times 5$

volume = $\underline{\underline{25}}$