Lecture 24

26 Oct 2012

Last time: A few ways to check whether a series converges.

- If the series is geometric, with first term $= a$
  common ratio $= r$

  then if $|r| < 1$, series converges
  $|r| \geq 1$, series diverges

- Test For Divergence: if $\lim_{n\to\infty} a_n$ doesn't exist, or it exists but it's $\neq 0$,
  then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex Does $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{2n^2+4n-3}{7n^2+6}\right)$ converge?

Apply TFD: $\displaystyle\lim_{n\to\infty} \ln\left(\frac{2n^2+4n-3}{7n^2+6}\right) = \ln\left(\frac{2}{7}\right) \neq 0$

so, the Σ diverges.

Another trick for dealing with (some) series:

Ex $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

not geometric; $\displaystyle\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$ so TFD is no help;

amazingly we can find the sum exactly: use "partial fractions"

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$1 = A(n+1) + Bn$

$n=-1 \longrightarrow 1 = -B$

$n=0 \longrightarrow 1 = A$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{\infty} a_n$

$a_1 = 1 - \frac{1}{2}$ $\qquad s_1 = a_1 = 1 - \frac{1}{2}$ $\qquad = 1 - \frac{1}{2}$

$a_2 = \frac{1}{2} - \frac{1}{3}$ $\qquad s_2 = a_1 + a_2 = 1 \cancel{- \frac{1}{2} + \frac{1}{2}} - \frac{1}{3}$ $\qquad = 1 - \frac{1}{3}$

$a_3 = \frac{1}{3} - \frac{1}{4}$ $\qquad s_3 = a_1 + a_2 + a_3 = 1 \cancel{- \frac{1}{2} + \frac{1}{2}} \cancel{- \frac{1}{3} + \frac{1}{3}} - \frac{1}{4}$ $\qquad = 1 - \frac{1}{4}$

$a_4 = \frac{1}{4} - \frac{1}{5}$ $\qquad s_4 = 1 \cancel{- \frac{1}{2} + \frac{1}{2}} \cancel{- \frac{1}{3} + \frac{1}{3}} \cancel{- \frac{1}{4} + \frac{1}{4}} - \frac{1}{5} = 1 - \frac{1}{5}$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$$S_k = 1 - \frac{1}{k+1} \qquad \lim_{k \to \infty} S_k = 1$$

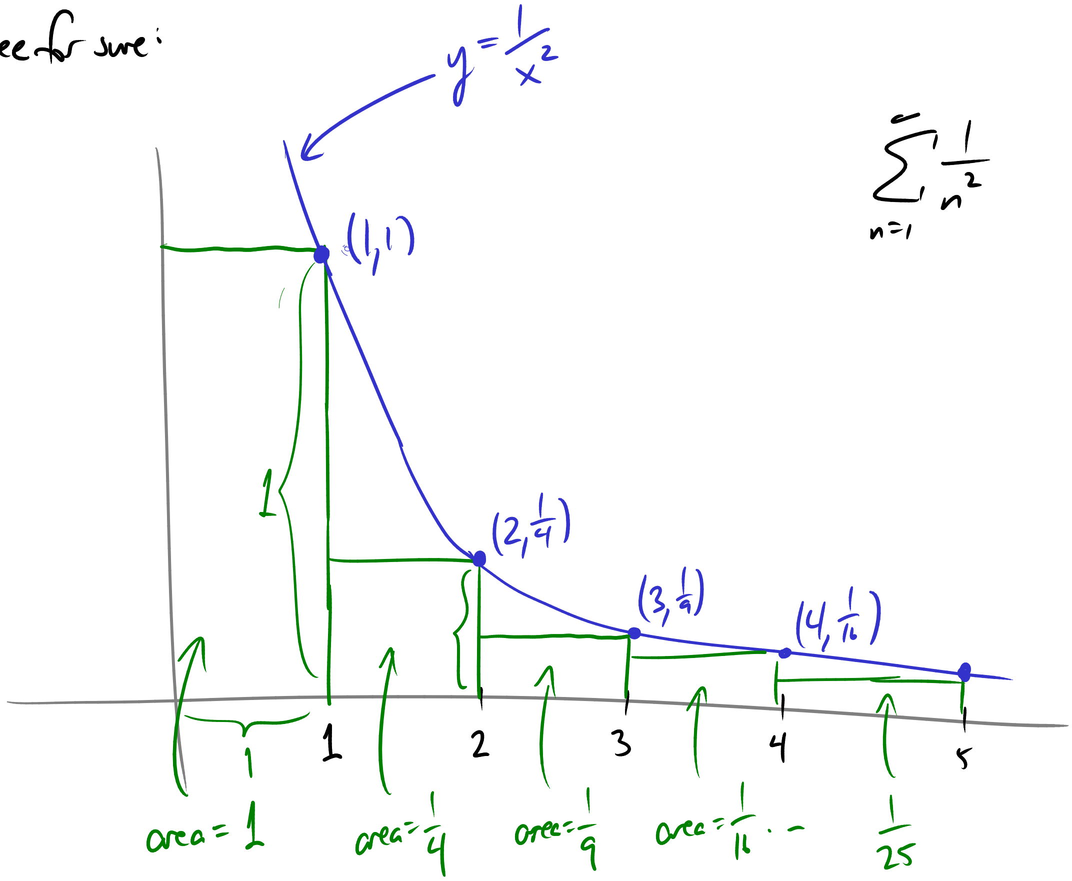
So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \underline{\underline{1}}$ ("telescoping series")

<u>Integral Test</u> (Ch 11.3)

Take the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

<u>Not</u> geometric; and $\displaystyle\lim_{n \to \infty} \frac{1}{n^2} = 0$ so TFD gives no info...

To see for sure:



$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

The picture shows that
$$\int_1^{\infty} \frac{1}{x^2}\,dx > \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$\int_1^{\infty} \frac{1}{x^2}\,dx > \sum_{n=2}^{\infty} \frac{1}{n^2}$$

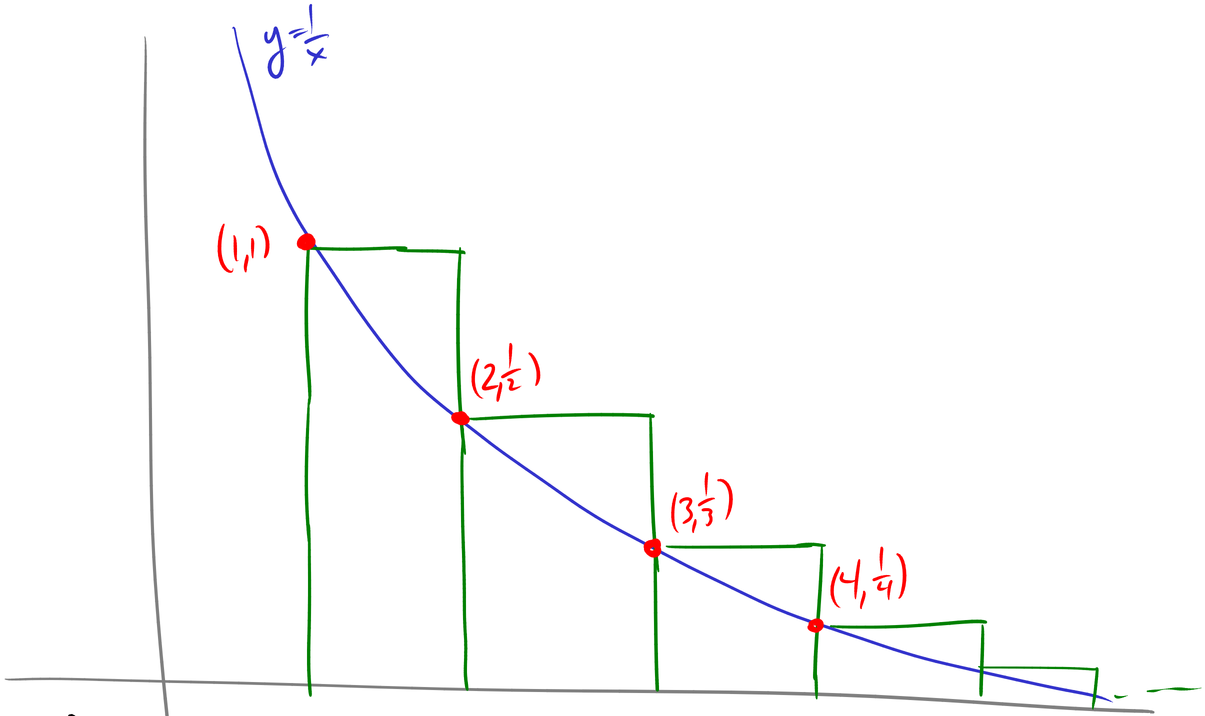
area under blue curve

sum of areas of rectangles

But $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges! So, $\sum_{n=2}^{\infty} \frac{1}{n^2}$ also converges.

So, $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$ also converges!

How about $\sum_{n=1}^{\infty} \frac{1}{n}$?



Total area of rectangles $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Area under blue curve $= \int_1^\infty \frac{1}{x}\,dx$

So $\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^\infty \frac{1}{x}\,dx$

And $\int_1^\infty \frac{1}{x}\,dx$ diverges (to ∞) So $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges (to ∞)

General rule (Integral Test):

Suppose $f(x)$ is a continuous, decreasing (positive) function defined for $1 \le x < \infty$. Say $a_n = f(n)$.

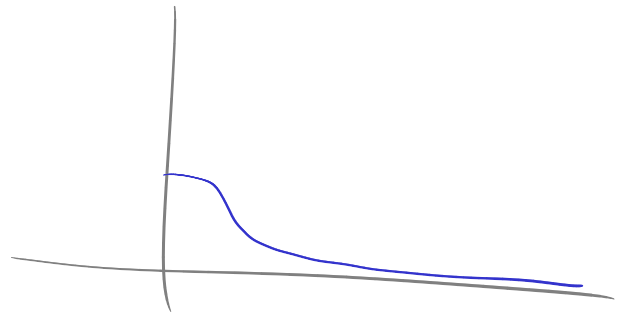
Then:

If $\int_1^\infty f(x)\,dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\int_1^\infty f(x)\,dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge?

$f(x) = \frac{1}{x^2+1}$ is decreasing



so look at $\int_1^\infty \frac{1}{x^2+1}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x \Big]_1^t$

$= \lim_{t\to\infty} \left(\tan^{-1}t - \frac{\pi}{4}\right)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, converges

So $\sum \frac{1}{n^2+1}$ converges by $\int$ test.