Lecture 30

9 Nov 2012

Housekeeping: my office hours will now be MF 11am-12noon

Last time: Absolute and conditional convergence
Ratio Test

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\qquad a_n = \dfrac{n^n}{n!}$

Ratio Test: look at $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!}$$

$$= \frac{(n+1)\cdot(n+1)^n}{n^n} \cdot \frac{n!}{(n+1)!}$$

$$= (n+1)\cdot\left(\frac{n+1}{n}\right)^n \cdot \frac{n!}{(n+1)!}$$

$$= (n+1)\cdot\left(1+\frac{1}{n}\right)^n \cdot \frac{n!}{(n+1)!}$$

$$= \cancel{(n+1)}\cdot\left(1+\frac{1}{n}\right)^n \cdot \frac{1}{\cancel{n+1}}$$

$$= \left(1+\frac{1}{n}\right)^n$$

So $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Since $e>1$, Ratio Test says $\displaystyle\sum \frac{n^n}{n!}$ diverges.

Ex $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ $\qquad a_n = \frac{\sqrt{n}}{1+n^2}$

Suppose we try Ratio Test on this:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}/(1+(n+1)^2)}{\sqrt{n}/(1+n^2)} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}$$

and $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2} = 1$ (sketch of reason: $\sim \frac{n^{5/2}}{n^{5/2}}$)

So the Ratio Test is <u>inconclusive</u> here.

(Could see that this $\sum$ converges by Limit Comp Test, with $b_n = \frac{1}{n^{3/2}}$.)

Root Test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u>.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ (or $=\infty$) then $\sum_{n=1}^{\infty}$ is <u>divergent</u>.

[If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then the Root Test is <u>inconclusive</u>.]

Ex $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ $\qquad a_n = \left(\frac{2n+3}{3n+2}\right)^n$

Root Test: $\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

So $\sum \left(\frac{2n+3}{3n+2}\right)^n$ <u>converges absolutely</u>

Strategy for Testing Series (Ch 11.7)

Classify the series according to its form.

1) $\sum \frac{1}{n^p}$: p-test.

2) $\sum ar^{n-1}$ or $\sum ar^n$: geom. series $\begin{cases} \text{conv. if } |r|<1 \\ \text{div. if } |r|\geq 1 \end{cases}$

3) If the series looks similar to a p-series or geom series: try comparison or limit-comparison (picking b_n to be the p-series or geom series). (If the series has some negative terms, then apply this method instead to $\sum |a_n|$ — i.e. test for absolute convergence.)

4) If you can see easily that $\lim_{n\to\infty} a_n \neq 0$, use Test For Divergence.

5) If the series is of form $\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$ try Alternating Series Test.

6) If the series involves factorials, or other products with n terms, like k^n try Ratio Test.
[But not for series where a_n is just a rational function, like $\frac{5n^2+7}{6n^3+8}$, b/c Ratio Test will always be inconclusive for these]

7) If $a_n = (\text{something})^n$ try Root Test

8) If $a_n = f(n)$ and you know how to do $\int_1^\infty f(x)\,dx$
and $f(x)$ is decreasing for large x
try Integral Test.

Ex $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+4}{3n^2+7n}\right)^{3n}$

$$a_n = \left(\frac{n^2+4}{3n^2+7n}\right)^{3n}$$
$$= \left(\left(\frac{n^2+4}{3n^2+7n}\right)^{3}\right)^{n}$$

use Root Test — $\sqrt[n]{\left(\frac{n^2+4}{3n^2+7n}\right)^{3n}} = \left(\frac{n^2+4}{3n^2+7n}\right)^{3}$

and take $\lim\limits_{n\to\infty}$ ...

Ex $\displaystyle\sum \frac{n+8}{2n+1}$: use TFD

Ex $\displaystyle\sum n^2 e^{-n^3}$: use $\int$ Test with $f(x) = x^2 e^{-x^3}$

Ex $\displaystyle\sum \frac{2^k}{k!}$: use Ratio Test

Ex $\displaystyle\sum \frac{1}{2+3^n}$: use Comp or Lim-Comp with $b_n = \frac{1}{3^n}$

Ex $\displaystyle\sum n \sin\left(\frac{1}{n}\right)$:

(Idea: when x is very small, $\sin(x) \approx x$)

use TFD $\displaystyle\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$

Ex $\sum (-1)^n \frac{n^3}{n^4+1}$: use Alt. Series Test $\Longrightarrow$ converges

(and if we wanted to see whether it converges absolutely, we'd look at $\sum \frac{n^3}{n^4+1}$ which diverges by Lim-Comp to $\frac{1}{n}$; so the $\sum$ converges conditionally)

Remark: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ $\qquad$ $\sum_{n=1}^{\infty} \frac{1}{n^3} = ?$ Deep!!
