Lecture 17

26 Feb 2010

Last time: trigonometric $\int$s

Like $\int \sin^a\theta \cos^b\theta \, d\theta$

$\int \sec^a\theta \tan^b\theta \, d\theta$

One more example:

Ex $I = \int \sec^3 x \, dx$

Int. by parts: $u = \sec x$ $\quad v = \tan x$

$du = \sec x \tan x \, dx \quad dv = \sec^2 x \, dx$

$$I = \int u \, dv = uv - \int v \, du$$

$$= \sec x \tan x - \int \tan x \sec x \tan x \, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

Use $\tan^2 x = \sec^2 x - 1$

$$I = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$I = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx$$

$$I = \sec x \tan x + \ln|\sec x + \tan x| + C - I$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$I = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

Next method:

Trigonometric Substitution (Ch 8.3)

Ex $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = ?$

$\left[-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right]$

A clever substitution: $x = 3\sin\theta$ $\quad$ [since $-3 \le x \le 3$]

$dx = 3\cos\theta\, d\theta$

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta}$$

$$= 3\sqrt{\cos^2\theta}$$

(since $\cos\theta > 0$) $\quad = 3\cos\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{(3\sin\theta)^2}\cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\, d\theta$$

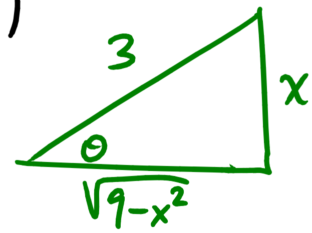
$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C$$

and remember that $x = 3\sin\theta$ $\qquad \theta = \sin^{-1}\left(\frac{x}{3}\right)$

To get $\cot\theta$ in terms of x:

$\frac{x}{3} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$



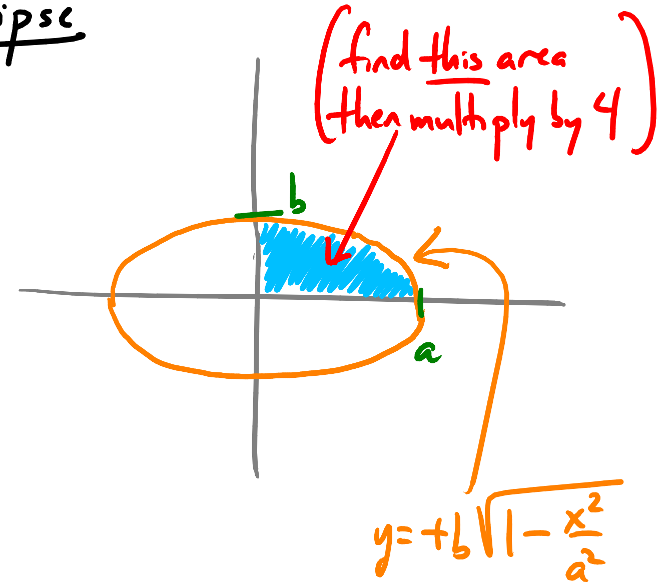
$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{9-x^2}}{x} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

So finally, the integral is

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Ex Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



solve for y: $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$A=4\int_0^a dx\; b\sqrt{1-\frac{x^2}{a^2}}$$

$$=4b\int_0^a dx\sqrt{1-\frac{x^2}{a^2}}$$

Want this to become

$\sqrt{1-\sin^2\theta}$

So, subst. $\frac{x^2}{a^2}=\sin^2\theta$

ie $x=a\sin\theta$

$dx=a\cos\theta\, d\theta$

Change limits: $x=0$ is $\theta=0$

$x=a$ is $\theta=\frac{\pi}{2}$

[because $\sin\theta=1$]

$$=4b\int_0^{\frac{\pi}{2}}a\cos\theta\, d\theta\cdot\sqrt{1-\sin^2\theta}$$

$$=4ba\int_0^{\frac{\pi}{2}}\cos\theta\, d\theta\sqrt{\cos^2\theta}$$

$$=4ba\int_0^{\frac{\pi}{2}}\cos\theta\, d\theta\,\cos\theta$$

$$=4ba\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$

$$= 4ba \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\ d\theta$$ ($\frac{1}{2}$-angle identity)

$$= 4ba \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2\theta\ d\theta$$

$$= \cdots$$

$$= \underline{\underline{ab\pi}}$$

Ex

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

Here use the identity $\tan^2+1=\sec^2$:

Substitute

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\, d\theta$$

$$\boxed{\begin{aligned}\sqrt{x^2+4} &= \sqrt{4\tan^2\theta+4}\\ &= 2\sqrt{\tan^2\theta+1}\\ &= 2\sqrt{\sec^2\theta}\\ &= 2\sec\theta\end{aligned}}$$

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx = \int \frac{1}{4\tan^2\theta\cdot 2\sec\theta}\cdot 2\sec^2\theta\, d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$:
$du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \frac{du}{u^2} = -\frac{1}{4}\left(\frac{1}{u}\right)+C$$

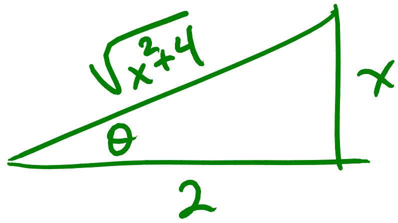
Subst. back to original variable:

$$u = \sin\theta$$
$$x = 2\tan\theta$$

What's u in terms of x?

$$\frac{x}{2} = \tan\theta$$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{x}{2}$



$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+4}}$

so $\quad u = \dfrac{x}{\sqrt{x^2+4}}$

and the integral is $\quad -\dfrac{1}{4u} = \underline{\underline{-\dfrac{1}{4}\dfrac{\sqrt{x^2+4}}{x} + C}}$

Table:

$\sqrt{a^2-x^2}$ use $x = a\sin\theta$, $1-\sin^2\theta = \cos^2\theta$

$\sqrt{a^2+x^2}$ use $x = a\tan\theta$, $1+\tan^2\theta = \sec^2\theta$

$\sqrt{x^2-a^2}$ use $x = a\sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$
