Lecture 18

1 March 2010

Notes are at http://www.ma.utexas.edu/users/neitzke

A difficult HW pb:

$$\int_0^a \sqrt{1+\sin\theta}\, d\theta$$

Could try: $\displaystyle\int_0^a \sqrt{1+\sin\theta}\,\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\,d\theta = \int_0^a \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}}\,d\theta = \int_0^a \frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}}$

$$u = 1-\sin\theta \qquad du = -\cos\theta\, d\theta \ldots$$

But this sometimes doesn't work because $1-\sin\theta$ could be zero somewhere in $0<\theta<a$. Then you'd get wrong answer!

Set $\theta = 2x$: $\sin\theta = \sin 2x = 2\sin x\cos x$

And use $1 = \cos^2 x + \sin^2 x$. Then $\sqrt{1+\sin\theta} = \sqrt{\cos^2 x+\sin^2 x+2\sin x\cos x}$

$(x = \frac{\theta}{2})$

$$= \sqrt{(\sin x+\cos x)^2}$$

$$= |\sin x+\cos x|$$

Last time: trig substitution

If you see $\sqrt{a^2-x^2}$ try $x = a\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad$ i.e. $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$\sqrt{a^2+x^2}$ try $x = a\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\quad \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$\sqrt{x^2-a^2}$ try $x = a\sec\theta$ $\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ $\quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$

Ex $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$ $\qquad$ try $x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$$= \int_0^{\pi/3} \frac{(4\sin\theta)^3}{\sqrt{16-(4\sin\theta)^2}} \cdot 4\cos\theta\,d\theta$$

limits: $x=0$ is $\sin\theta = 0$ ie $\theta = 0$

$x = 2\sqrt{3}$ is $\sin\theta = \frac{\sqrt{3}}{2}$ ie $\theta = \frac{\pi}{3}$

$$= \int_0^{\pi/3} \frac{4^3\sin^3\theta}{\sqrt{16(1-\sin^2\theta)}} \cdot 4\cos\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\sqrt{\cos^2\theta}} \cdot 4\cos\theta\,d\theta$$

$$= 4^3\int_0^{\pi/3} \sin^3\theta \cdot \frac{\cos\theta}{\cos\theta}\,d\theta$$

$$= 4^3\int_0^{\pi/3} \sin^3\theta\,d\theta$$

$$= 4^3\int_0^{\pi/3} \sin^2\theta \cdot (\sin\theta\,d\theta)$$

Want $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$= 4^3\int_0^{\pi/3} (1-\cos^2\theta)(\sin\theta\,d\theta) = 4^3\int_1^{1/2} (1-u^2)(-du) = \cdots = \underline{\underline{\frac{40}{3}}}$$

$\boxed{\sin^2\theta + \cos^2\theta = 1}$

Ex $\displaystyle\int \frac{dx}{\sqrt{x^2+8x+25}}$

Want to relate this to s.t. like $\displaystyle\int \frac{1}{\sqrt{u^2+a^2}}$

Complete the square: $u = x + c$ for some constant c

$$u^2 = x^2 + 2c\cdot x + c^2$$

take $c=4$, ie $u = x+4$, then $u^2 = x^2+8x+16$

So $u^2+9 = x^2+8x+25$ $\qquad du = dx$

Then $\displaystyle\int \frac{dx}{\sqrt{x^2+8x+25}} = \int \frac{du}{\sqrt{u^2+9}}$

So we can substitute $u = 3\tan\theta$ $\qquad du = 3\sec^2\theta\, d\theta$

Then $\displaystyle\int = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9\tan^2\theta+9}}$

$$= \int \frac{3\sec^2\theta}{3\sqrt{\tan^2\theta+1}}\, d\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$\left(= \int \frac{3\sec^2\theta}{3\sec\theta}\, d\theta\right)$$

$$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Write it in terms of x: use $u = 3\tan\theta \to \tan\theta = \frac{u}{3} = \frac{x+4}{3}$

To get $\sec\theta$ in terms of x, could draw $\triangle$ and use SOHCAHTOA.

But we know $\sec\theta = \frac{1}{3}\sqrt{x^2+8x+25}$!

So altogether $\displaystyle\int \frac{dx}{\sqrt{x^2+8x+25}} = \underline{\underline{\ln\left|\frac{1}{3}\sqrt{x^2+8x+25} + \frac{1}{3}(x+4)\right|}}$

Ex $\displaystyle\int x\sqrt{1-x^4}\,dx$

$u = x^2 \qquad du = 2x\,dx$

$u^2 = x^4 \qquad \frac{1}{2}du = x\,dx$

$$= \int \sqrt{1-u^2}\,(x\,dx)$$

$$= \frac{1}{2}\int \sqrt{1-u^2}\,du$$

$u = \sin\theta \qquad x^2 = u$

$du = \cos\theta\,d\theta$

$$= \frac{1}{2}\int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

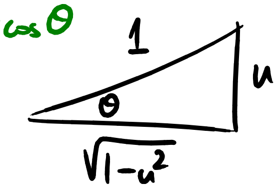
$$= \frac{1}{2}\int \cos^2\theta\,d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$



$$= \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + C$$

$\sin\theta = u$

$\cos\theta = \sqrt{1-u^2}$

$$= \frac{1}{4}\sin^{-1}(u) + \frac{1}{4}u\sqrt{1-u^2} + C$$

$$= \underline{\underline{\frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1-x^4} + C}}$$