Lecture 21

8 Mar 2010

Indeterminate forms and L'Hospital's Rule (Ch 7.8)

$$F(x) = \frac{x^2+2}{x^2-4}$$

What's $\lim_{x\to 1} F(x)$? Both num. and denom. stay finite (and nonzero): $\lim_{x\to 1} F(x) = \frac{1^2+2}{1^2-4} = \frac{3}{-3} = \underline{\underline{-1}}$

$\lim_{x\to 2} F(x)$? Numerator $\to 2^2+2 = 6$

Denominator $\to 2^2-4 = 0$

So we're getting $\frac{6}{0}$ here...

If $x \to 2$ from the <u>positive</u> direction, then we have $\frac{6}{(\text{very small positive})}$

so $\lim_{x\to 2^+} F(x) = +\infty$

If $x \to 2$ from the <u>negative</u> direction we have $\frac{6}{(\text{very small negative})}$

So $\lim_{x\to 2^-} F(x) = -\infty$

But what about more complicated cases where we get...

$\frac{0}{0}$, or $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞, ...

How do we deal with these "indeterminate forms"?

Sometimes just algebraic manipulations:

<u>Ex</u> $$\lim_{x\to\infty} \frac{x^2+3x+4}{3x^2-6x-7}$$

If we take the limits of num, denom separately, get $\frac{\infty}{\infty}$.

But we can <u>divide</u> both num, denom by x^2:

then get $$\lim_{x\to\infty} \frac{(x^2+3x+4)\cdot\left(\frac{1}{x^2}\right)}{(3x^2-6x-7)\cdot\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{4}{x^2}}{3-\frac{6}{x}-\frac{7}{x^2}}$$

Now take lim of num, denom: get

$$= \frac{\lim_{x\to\infty} 1+\frac{3}{x}+\frac{4}{x^2}}{\lim_{x\to\infty} 3-\frac{6}{x}-\frac{7}{x^2}} = \frac{1}{3}$$

Sometimes those kinds of methods don't help: use <u>L'Hospital's Rule</u>

<u>L's rule</u>: If $\left[\begin{array}{c} \lim_{x\to a} f(x) = \pm\infty \\ \underline{\text{and}} \\ \lim_{x\to a} g(x) = \pm\infty \end{array}\right]$ <u>OR</u> $\left[\begin{array}{c} \lim_{x\to a} f(x) = 0 \\ \underline{\text{and}} \\ \lim_{x\to a} g(x) = 0 \end{array}\right]$

THEN $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Ex $\lim_{t\to 0} \frac{e^t-1}{t}$ Try $\frac{\lim_{t\to 0} e^t-1}{\lim_{t\to 0} t}$: that's $\frac{0}{0}$

so need L'H rule:

$$\lim_{t\to 0} \frac{e^t-1}{t} = \lim_{t\to 0} \frac{\frac{d}{dt}(e^t-1)}{\frac{d}{dt}(t)} = \lim_{t\to 0} \frac{e^t}{1} = \frac{1}{1} = \underline{\underline{1}}$$

Ex $\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}$ $\frac{\lim_{x\to 0} \sin 4x}{\lim_{x\to 0} \tan 5x} = \frac{0}{0}$ so need L'H:

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\sin 4x}{\frac{d}{dx}\tan 5x} = \lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \frac{4\cdot 1}{5\cdot 1} = \underline{\underline{\frac{4}{5}}}$$

Ex $\lim_{x\to\infty} \frac{e^x}{x^2}$: $\frac{\infty}{\infty}$ so need L'H rule

$= \lim_{x\to\infty} \frac{e^x}{2x}$: $\frac{\infty}{\infty}$ use L'H again

$= \lim_{x\to\infty} \frac{e^x}{2}$: $\frac{\infty}{2}$, ie $\underline{\underline{\lim_{x\to\infty} \frac{e^x}{x^2} = \infty}}$

Ex $\lim_{x\to\pi} \frac{\sin x}{1-\cos x} = \frac{0}{2} = \underline{\underline{0}}$ (No L'H rule here!)

To deal with $0 \cdot \infty$:

Try to rewrite it as $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Ex $\lim_{x \to 0^+} x \ln x$

$\left[\begin{array}{l} \lim_{x \to 0^+} x = 0 \\ \lim_{x \to 0^+} \ln x = -\infty \end{array} \right.$ so this is $0 \cdot (-\infty)$]

$$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}$$ [this is $\frac{\infty}{\infty}$ so can use L'H rule]

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to 0^+} (-x) = \underline{\underline{0}}$$

Ex $\lim_{x \to \infty} x \sin\left(\frac{\pi}{x}\right)$ looks like $\infty \cdot 0$: rewrite it

$$= \lim_{x \to \infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\left(\frac{1}{x}\right)}$$ [$\to \frac{0}{0}$ so use L'H]

$$= \lim_{x \to \infty} \frac{\left(-\pi/x^2\right)\left(\cos \frac{\pi}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to \infty} \pi \cos\left(\frac{\pi}{x}\right) = \pi \cos(0) = \underline{\underline{\pi}}$$

For $\infty - \infty$:

again try to make it into $\frac{\infty}{\infty}$ or $\frac{0}{0}$

Ex $\lim_{x \to (\pi/2)^-} \sec x - \tan x \qquad (\to \infty - \infty)$

$= \lim_{x \to (\frac{\pi}{2})^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x}$

$= \lim_{x \to (\frac{\pi}{2})^-} \frac{1 - \sin x}{\cos x} \to \frac{0}{0}$ since L'H:

$= \lim_{x \to (\frac{\pi}{2})^-} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = \underline{\underline{0}}$