Lecture 22

10 Mar 2010

Housekeeping:
- Exam 2 Tue Apr 6 7-9pm WEL 1.316
- HW 9 due Tue Mar 23 3am

Last time: Indeterminate forms and L'Hospital's rule

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$ we discussed how to do

But what about 1^∞, 0^0, ∞^0?

Basic strategy: take the $\log$ and evaluate its limit

$$\lim f(x) = e^{\lim \log f(x)}$$

Ex: $\displaystyle\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$ $\left[\longrightarrow 1^\infty\right]$

$$= e^{\lim_{x\to\infty} \log\left(1+\frac{a}{x}\right)^{bx}}$$

and
$$\lim_{x\to\infty} \log\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} bx \log\left(1+\frac{a}{x}\right) \quad \left[\longrightarrow \infty\cdot 0\right]$$

$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{a}{x}\right)}{\left(\frac{1}{bx}\right)} \quad \left[\longrightarrow \frac{0}{0}\right]$$

use L'H:

$$= \lim_{x\to\infty} \frac{\left(-\frac{a}{x^2}\right)\cdot\left(\frac{1}{1+\frac{a}{x}}\right)}{\left(-\frac{1}{bx^2}\right)}$$

$$= \lim_{x\to\infty} \frac{ab}{1+\frac{a}{x}} = \frac{ab}{1} = ab$$

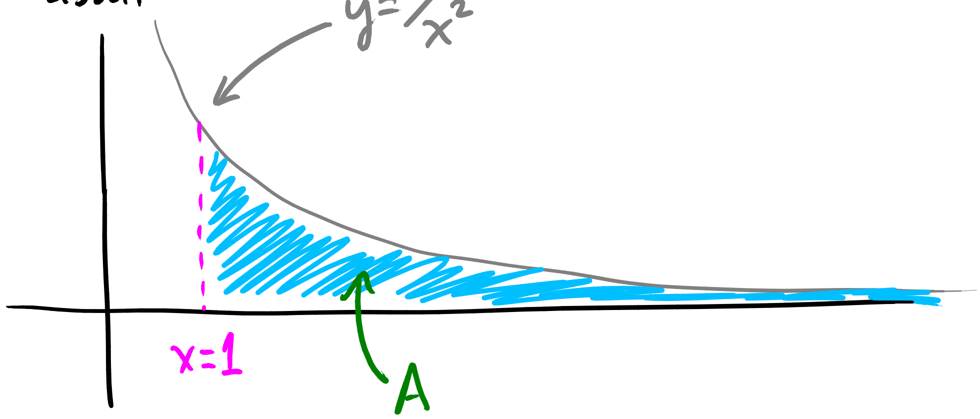
So $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$

Improper Integrals (Ch 8.8)

The $\int$ we did so far, $\int_a^b f(x)\,dx$ where $f(x)$ was well defined, finite for all $a \leq x \leq b$.

Those are called proper (definite) integrals.

How about:



the area of this infinite region?

$$A = \int_1^\infty \frac{1}{x^2}\,dx$$

This really means: define $A(t) = \int_1^t \frac{1}{x^2}\,dx$

then $A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx.$

So, let's calculate it:

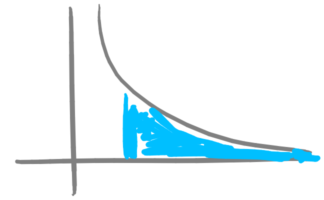
$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x} \Big|_1^t \right) = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = \underline{\underline{1}}$$

$$\underline{\underline{A = 1}}$$

When the limit exists (like it did here), we call the improper integral <u>convergent</u>.

$$\left[\int_1^\infty \frac{1}{x^2}\,dx \quad \text{is } \underline{\text{convergent}}. \right]$$

How about $\displaystyle\int_1^\infty \frac{1}{x}\,dx$?



$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\left[\left(\ln x\,\Big|_1^t\right)\right]$$

$$= \lim_{t\to\infty} (\ln t)$$

This limit doesn't exist (goes to $+\infty$)

This improper integral is not convergent — it is divergent.

[If the limit is $+\infty$ or $-\infty$, or doesn't exist, call it divergent]

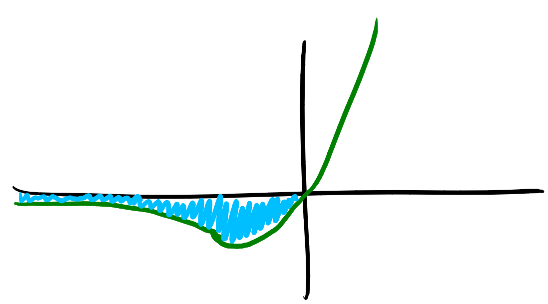
So far we saw: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent

$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent

General rule:

$$\int^\infty \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent for } p>1 \\ \text{divergent for } p\le 1 \end{cases}$$

Ex $\displaystyle\int_{-\infty}^0 x\,e^x\,dx$



$$= \lim_{t\to-\infty} \int_t^0 x e^x\,dx$$

(integrate by parts)

$$= \lim_{t\to-\infty}\left[-\underbrace{te^t}_{\infty\cdot 0} - 1 + \underbrace{e^t}_{\to 0}\right]$$

use L'H rule for the 1st term: $\displaystyle\lim_{t\to-\infty}\left(-te^{t}\right) = \lim_{t\to-\infty}\frac{-t}{e^{-t}} \quad \frac{\infty}{\infty}$

FALSE START:

$$\left[\ = \lim_{t\to-\infty}\frac{e^{t}}{\left(-\frac{1}{t}\right)} \quad \frac{0}{0}\right.$$

$$\left.(L'H) \quad = \lim_{t\to-\infty}\frac{e^{t}}{\frac{1}{t^2}}\ \right]$$

$$(L'H) \quad = \lim_{t\to-\infty}\frac{-1}{-e^{-t}}$$

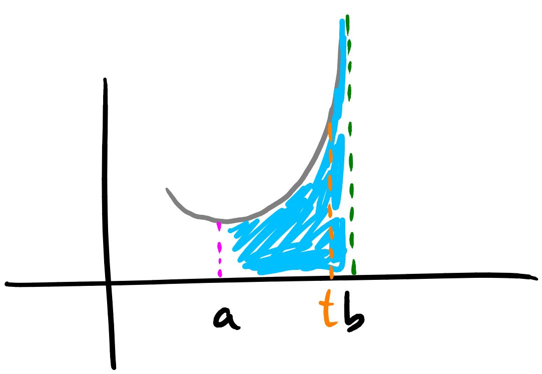
$$= \lim_{t\to-\infty}\frac{1}{e^{-t}} = \frac{1}{\infty} = \underline{\underline{0}}$$

So altogether $\displaystyle\int_{-\infty}^{0} xe^{-x}\,dx = \underline{\underline{-1}}$ (<u>convergent</u>)

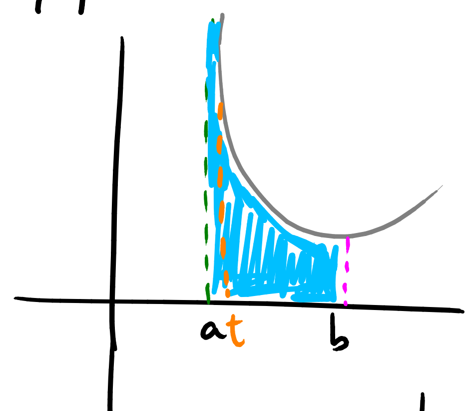
Another kind of improper integral:

$\displaystyle\int_a^b f(x)\,dx$ where $f(x)$ becomes infinite somewhere

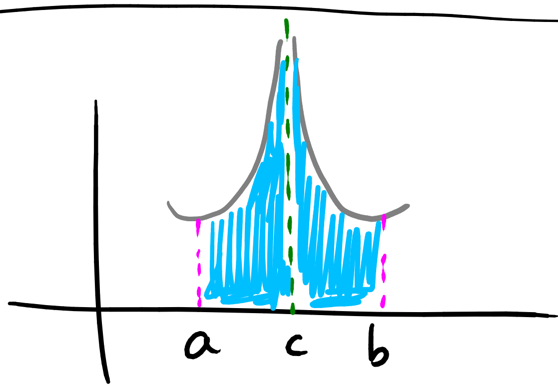
[= $f(x)$ has a vertical asymptote]



Here $\displaystyle\int_a^b f(x)\,dx$ <u>means</u> $\displaystyle\lim_{t\to b^-}\int_a^t f(x)\,dx$



Here $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$



Here

$\displaystyle\int_a^b f(x)\,dx$ <u>means</u>

$$\lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$