Lecture 24

22 Mar 2010

Housekeeping: Exam 2 Tue Apr 6 7-9pm WEL 1.316

Last time: improper integrals

So far we always looked at functions $f(x)$

only one variable.

In reality things often depend on more than one variable.

e.g. the heat index depends on temperature and humidity.

$f(x,y)$ x y

or the elevation depends on longitude and latitude

the price of cheese depends on supply and demand

Partial derivatives (Ch 15.3)

If we have $f(x,y)$ we can ask:

- How does f change if we vary x a little while holding y constant?

Define $$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

[To calculate it, just treat y as const and differentiate like usual with resp to x.]

Ex If $f(x,y) = x^2 \sin(y)$ then $\frac{\partial f}{\partial x} = 2x\sin(y)$

Ex If $f(x,y) = x^2 \sin(y)$ then $\frac{\partial f}{\partial x}$ at $(x,y) = (1, \pi/2)$ is $2 \cdot 1 \cdot 1 = 2$

Similarly $\dfrac{\partial f}{\partial y} = \lim_{h \to 0} \dfrac{f(x, y+h) - f(x,y)}{h}$

(hold x constant and differentiate w/respect to y).

<u>Ex</u> Say $f(x,y) = 4x^2y + 7\sin(x)$

$$\frac{\partial f}{\partial x} = \underline{\underline{8xy + 7\cos(x)}} \qquad \frac{\partial f}{\partial y} = \underline{\underline{4x^2}}$$

Can also look at <u>2nd deriv</u>:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \underline{\underline{8y - 7\sin(x)}} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underline{\underline{8x}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(4x^2) = \underline{\underline{8x}} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(4x^2) = \underline{\underline{0}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

<u>Notice</u>: $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$! This always happens (whenever both are continuous)

Ex $f(x,y) = \sin(xy)$

$$\frac{\partial f}{\partial x} = y\cos(xy) \qquad \frac{\partial f}{\partial y} = x\cos(xy)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(y\cos(xy)\right) = y\cdot y(-\sin(xy)) = -y^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(x\cos(xy)\right) = x\cdot x(-\sin(xy)) = -x^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\cos(xy)\right) = 1\cdot\cos(xy) + x\cdot(-y\sin(xy))$$

$$= \underline{\underline{\cos(xy) - xy\sin(xy)}}$$

[Also, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cos(xy) - xy\sin(xy)$]

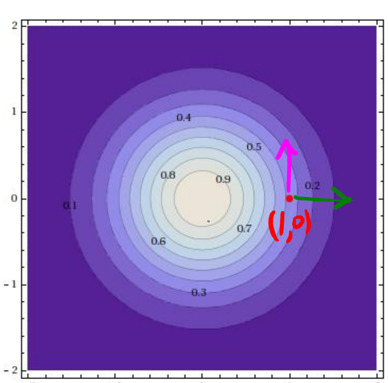
because $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$

Picturing the partial deriv:

Say $f = e^{-x^2-y^2}$ $\qquad \frac{\partial f}{\partial x} = -2x\,e^{-x^2-y^2} \qquad \frac{\partial f}{\partial y} = -2y\,e^{-x^2-y^2}$

So at $(x,y) = (1,0)$: $\frac{\partial f}{\partial x} = -\frac{2}{e} < 0 \qquad \frac{\partial f}{\partial y} = 0$

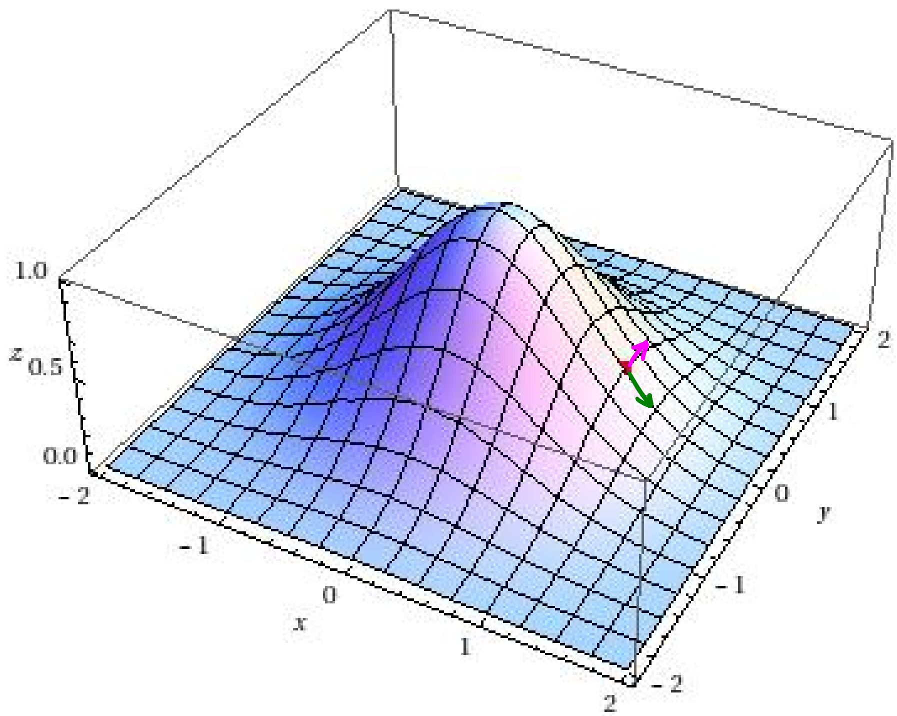
See this from the "contour map" of f:
(lighter color = larger value of f)



$\longrightarrow$: going downhill
i.e. f decreasing
i.e. $\frac{\partial f}{\partial x} < 0$

$\uparrow$: neither uphill nor downhill
i.e. $\frac{\partial f}{\partial y} = 0$

3-d picture of f: (height $= f(x,y)$)



$\rightarrow$ downhill

$\uparrow$ neither uphill nor downhill

Notation: Write f_x for $\dfrac{\partial f}{\partial x}$

f_y for $\dfrac{\partial f}{\partial y}$

f_{xx} for $\dfrac{\partial^2 f}{\partial x^2}$

f_{xy} for $\dfrac{\partial^2 f}{\partial x \partial y}$ [so $f_{xy} = f_{yx}$]

f_{yx} for $\dfrac{\partial^2 f}{\partial y \partial x}$

f_{yy} for $\dfrac{\partial^2 f}{\partial y^2}$

[etc. for higher derivatives]