Lecture 25 | 24 Mar 2010

Last time: partial derivatives

$$\begin{tikzcd}
 & & \frac{\partial^2 f}{\partial x^2} \\
f(x,y) \arrow[r, squiggly] \arrow[dr, squiggly] & \frac{\partial f}{\partial x} \arrow[ur, squiggly] \arrow[r, squiggly] & \frac{\partial^2 f}{\partial x \partial y} \\
 & \frac{\partial f}{\partial y} \arrow[r, squiggly] \arrow[dr, squiggly] & \frac{\partial^2 f}{\partial y \partial x} \\
 & & \frac{\partial^2 f}{\partial y^2}
\end{tikzcd}$$

these two are equal

Notation:

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial f}{\partial y} = f_y$$

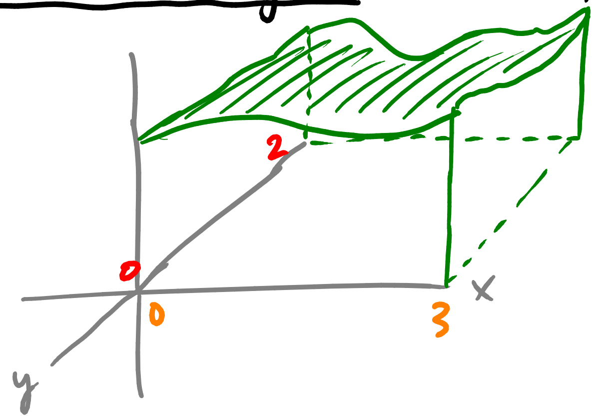
$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \quad \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$f_{xy} = f_{yx}$$

Terminology: If $f(x,y) =$ productivity
where $x =$ capital
$y =$ labor

Then $f_x$ is called marginal productivity of capital
$f_y$ is " " of labor

# Iterated integrals (Ch 16.2)




<u>Surface</u> $z=f(x,y)$ [like $y=f(x)$]

In 1-variable calc, we studied <u>area</u> under the <u>curve</u> $y=f(x)$

Now study <u>volume</u> under the <u>surface</u>

$z=f(x,y)$

cross section area

Cut by planes at fixed $y$: $V=\int_0^2 A(y)\,dy$

$A(y)=\int_0^3 f(x,y)\,dx$ gives the cross section [as usual]

So $V=\int_0^2\left[\int_0^3 f(x,y)\,dx\right]dy$

<u>Ex</u> Suppose $f(x,y)=4xy+3x^2$

Then $V=\int_0^2\left[\int_0^3 4xy+3x^2\,dx\right]dy$

$$=\int_0^2\left[2x^2y+x^3\Big|_{x=0}^{x=3}\right]dy$$

$$=\int_0^2(18y+27-0)\,dy$$

$$=9y^2+27y\Big|_{y=0}^{y=2}$$

$$=36+54=\underline{\underline{90}}$$

We could also try doing the $\int$'s in the <u>other order</u>:

$$V = \int_0^3 \left[ \int_0^2 4xy + 3x^2 \, dy \right] dx$$

$$= \int_0^3 \left[ 2xy^2 + 3x^2 y \Big|_{y=0}^{y=2} \right] dx$$

$$= \int_0^3 8x + 6x^2 \, dx$$

$$= 4x^2 + 2x^3 \Big|_{x=0}^{x=3}$$

$$= 36 + 54 = \underline{\underline{90}}$$

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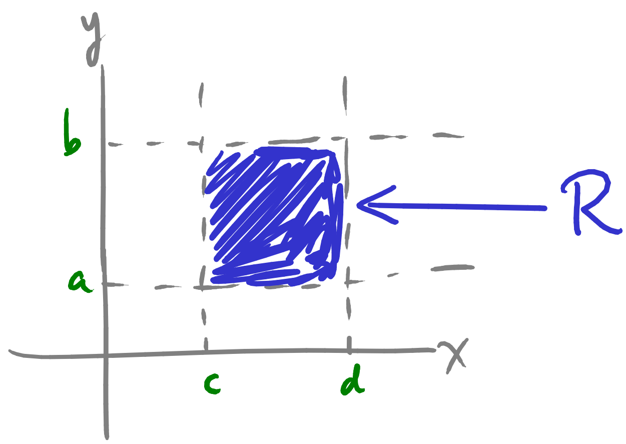
Notice, answer doesn't depend on the order:

$$\int_a^b \left[ \int_c^d f(x,y) \, dx \right] dy = \int_c^d \left[ \int_a^b f(x,y) \, dy \right] dx$$

("Fubini's Theorem")

We also write it as

$$\iint_R f(x,y) \, dA$$



Ex If $R = \{1 \le x \le 2,\ 0 \le y \le \pi\}$ and $f(x,y) = y\sin(xy)$

What is $\iint_R f(x,y)\, dA$ ?

It is $\int_0^\pi \left[ \int_1^2 y\sin(xy)\, dx \right] dy$

(or: $\int_1^2 \left[ \int_0^\pi y\sin(xy)\, dy \right] dx$, but that's harder to calculate)

$$= \int_0^\pi \left( -\cos(xy) \Big|_{x=1}^{x=2} \right) dy$$

$$= \int_0^\pi \left( -\cos(2y) + \cos(y) \right) dy$$

$$= -\frac{1}{2}\sin(2y) + \sin(y) \Big|_{y=0}^{y=\pi}$$

$$= \underline{\underline{0}}$$

Ex Find the volume of the solid which lies under the graph of

$$z = f(x,y) = 4 + x^2 - y^2$$

and over the rectangle $\left.\begin{array}{l} -1 \leq x \leq 1 \\ 0 \leq y \leq 2 \end{array}\right]$ (call this rectangle R)

$$V = \iint_R f(x,y)\, dA$$

$$= \iint_R 4 + x^2 - y^2 \, dA$$

$$= \int_{-1}^{1} \left[ \int_0^2 4 + x^2 - y^2 \, dy \right] dx$$

$$= \int_{-1}^{1} \left[ 4y + x^2 y - \frac{1}{3} y^3 \Big|_{y=0}^{y=2} \right] dx$$

$$= \int_{-1}^{1} \left[ \left(8 + 2x^2 - \frac{8}{3}\right) - 0 \right] dx$$

$$= \int_{-1}^{1} \frac{16}{3} + 2x^2 \, dx$$

$$= \frac{16}{3} x + \frac{2}{3} x^3 \Big|_{-1}^{1} = 6 + 6 = \underline{\underline{12}}$$