Lecture 29

2 Apr 2010

Housekeeping:

Exam 2 7–9pm April 6 (Tue)

$$\Big[\underbrace{7.8}_{\text{indeterm. forms and L'H}},\ \underbrace{8.1\text{–}8.5,\ 8.8}_{\text{integrals}},\ \underbrace{15.3,\ 16.2\text{–}16.3}_{\text{multivar.}},\ \underbrace{12.1}_{\text{seq.}}\Big]$$

Review session 7–9pm April 5 (Mon)

Last time: Sequences — ordered list of #'s

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

ex. $a_n = n^2(-1)^n$: $-1, 4, -9, 16, -25, \ldots$

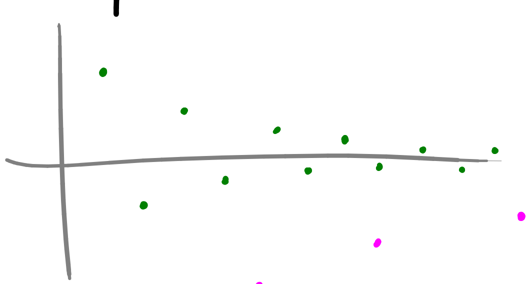
$a_n = \frac{n}{n+1}$: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$a_n = 5$: $5, 5, 5, 5, \ldots$

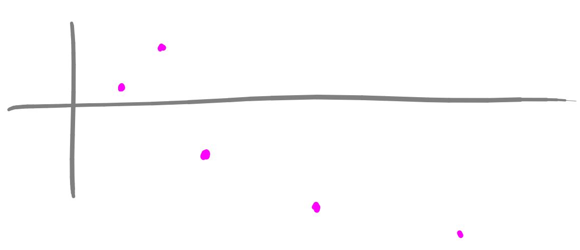
$a_n =$ the n^{th} Fibonacci # : $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$$a_n = a_{n-2} + a_{n-1}$$

Main question we ask about sequences: do they converge as $n \to \infty$?



$\lim_{n\to\infty} a_n = 0$



a_n diverges

A useful fact: $$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Ex Does the sequence $a_n = \left(1-\frac{7}{n}\right)^{-2n}$ converge? If so, to what?

$$\lim_{n\to\infty}\left(1-\frac{7}{n}\right)^{-2n}$$

$$=\left[\lim_{n\to\infty}\left(1-\frac{7}{n}\right)^{n}\right]^{-2}$$

$$=\left[e^{-7}\right]^{-2} = e^{14} \quad \text{(convergent)}$$

Ex Does $a_n = 2+\frac{1}{n}+\frac{n!+1}{(n+1)!}$ converge? (To what?)

$$= \underset{\substack{\downarrow\\2}}{2}+\underset{\substack{\downarrow\\0}}{\frac{1}{n}}+\boxed{\frac{n!}{(n+1)!}}+\underset{\substack{\downarrow\\0}}{\frac{1}{(n+1)!}}$$

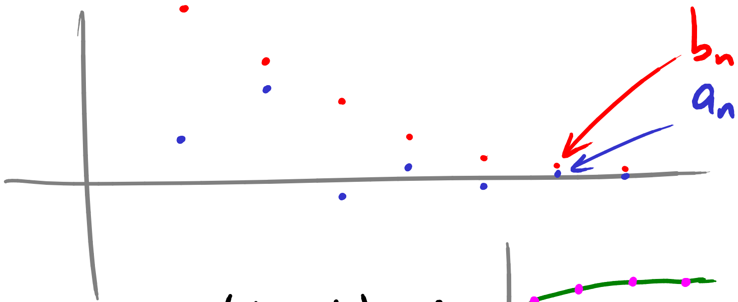
[because $(n+1)! \to \infty$]

factorial: $k!$ means $1\cdot2\cdot3\cdot(\cdots)\cdot k$ (e.g. $4! = 1\cdot2\cdot3\cdot4 = 24$)

to deal with this term, write it as
$$\frac{1\cdot2\cdot3\cdot\dots\cdot n}{1\cdot2\cdot3\cdot\dots\cdot n\cdot(n+1)} = \frac{1}{n+1} \longrightarrow 0$$

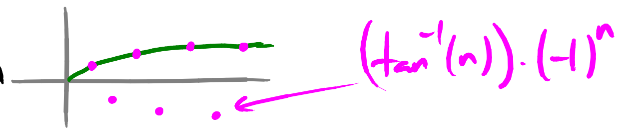
so $\lim_{n\to\infty} a_n = 2$

Another useful fact: if $|a_n| < b_n$ and $b_n \to 0$



then $a_n \to 0$ also!

("Squeeze Theorem")



<u>Ex</u> $\quad a_n = \dfrac{(\tan^{-1} n)\cdot(-1)^n}{\sqrt{n}}$

has $\lim\limits_{n\to\infty} a_n = 0$: <u>intuitively</u> because the <u>top</u> stays bounded as $n \to \infty$, but the <u>bottom</u> goes $\to \infty$ as $n \to \infty$.

Or, to <u>prove</u> it: $\left|(\tan^{-1} n)\cdot(-1)^n\right| = \left|\tan^{-1} n\right| \leq \dfrac{\pi}{2}$

so $\quad |a_n| \leq \dfrac{\pi/2}{\sqrt{n}}$

Call $b_n = \dfrac{\pi}{2}\cdot\dfrac{1}{\sqrt{n}}$. $\quad \lim\limits_{n\to\infty} b_n = 0$

<u>and</u> $|a_n| \leq b_n$

so $\lim\limits_{n\to\infty} a_n = 0$

Ex $\lim_{n\to\infty} \frac{n!}{n^n} = ?$

Rewrite it as $\frac{1\cdot 2\cdot 3\cdot 4 \cdot \cdots \cdot n}{\underbrace{n\cdot n\cdot n\cdot n\cdot \cdots \cdot n}_{n \text{ times}}}$ e.g. $4^4 = 4\cdot 4\cdot 4\cdot 4$

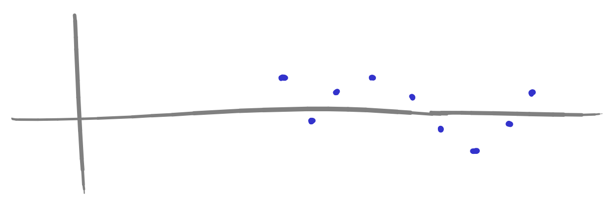
$= \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdot \cdots \cdot \frac{n}{n}$

$= \frac{1}{n}\cdot\left(\text{something} < 1\right)$

$< \frac{1}{n} \longrightarrow 0$

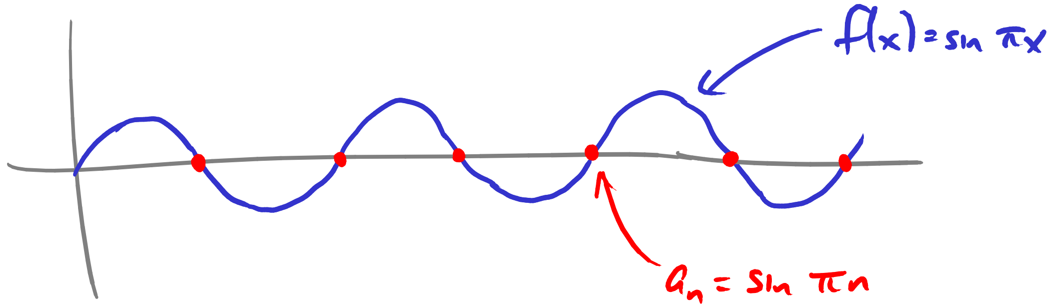
So $\lim_{n\to\infty} \frac{n!}{n^n} = \underline{\underline{0}}$

$a_n = \frac{n^2\cos(n)}{1+n^2}$. Intuition: $\sim \cos(n)$ for big n



$\longrightarrow$ <u>divergent</u>

$a_n = \sin(\pi n)$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = \underline{\underline{0}}$



NB: $\{a_n\}$ has a limit as $n \to \infty$
even though $f(x) = \sin(\pi x)$ doesn't have a limit as $x \to \infty$!

$$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos(0) = \underline{\underline{1}}$$

$$\lim_{n\to\infty} \ln(2n^2+1) - \ln(n^2+1) = \lim_{n\to\infty} \ln\frac{2n^2+1}{n^2+1} = \underline{\underline{\ln 2}}$$

Fact:

$r=2$: $2, 4, 8, 16, \ldots$ divergent

$r=\frac{1}{2}$: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ convergent ($\to 0$)

$$\lim_{n\to\infty} r^n = \begin{cases} 0 \text{ if } -1<r<1 \\ 1 \text{ if } r=1 \\ \text{does not exist (divergent) if } |r|>1 \end{cases}$$
