Lecture 35

16 Apr 2010

What do we do with a series like:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

This is an example of an <u>alternating</u> <u>series</u>

(so called because the signs of the terms alternate: $+, -, +, -, +, -, \ldots$)

(could also have $-, +, -, +, -, +, \ldots$)

<u>Alternating Series Test</u>

Suppose we have the series $\sum_{n=1}^{\infty} (-1)^n b_n$ $\left[\text{or } \sum_{n=1}^{\infty} (-1)^{n+1} b_n\right]$

where all $b_n > 0$.

If the b_n are <u>decreasing</u> ($b_{n+1} \leq b_n$ for all n)

and $\lim_{n\to\infty} b_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^n b_n$ <u>converges</u>. $\left(\text{or } \sum_{n=1}^{\infty} (-1)^{n+1} b_n\right)$

<u>Ex</u> $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

This is alternating series with $b_n = \frac{1}{n}$.

Try applying Alt. Series Test:

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓
- Are b_n decreasing? $b_{n+1} = \frac{1}{n+1}$, $b_n = \frac{1}{n}$, and $\frac{1}{n+1} \leq \frac{1}{n}$

So $b_{n+1} \le b_n$

So Alt. Series Test says $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges.

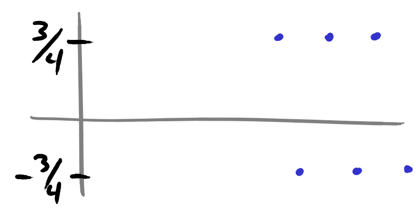
Ex $\quad \sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1} = \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{3n}{4n-1}\right)$

alternating with $b_n = \frac{3n}{4n-1} > 0$

We could try Alt. Series Test:

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$ so the test doesn't apply.

In fact, write $a_n = (-1)^n \left(\frac{3n}{4n-1}\right)$, $\lim_{n\to\infty} a_n$ does not exist (oscillates)



So $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$ diverges by the "Test For Divergence"

Ex $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

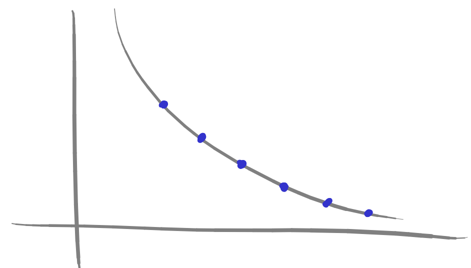
Alternating series: $b_n = \dfrac{n^2}{n^3+1}$

Use Alt. Series Test –

- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$

- Is $b_{n+1} \leq b_n$?

  Look at $f(x) = \dfrac{x^2}{x^3+1}$



  Take $f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2}$ (Quotient Rule + simplific.)

For large enough x,

positive

negative

$$\frac{x(2-x^3)}{(x^3+1)^2}$$

positive

So $f'(x) < 0$ for large enough x

So b_n is decreasing

So Alt. Series Test applies; so $\displaystyle\sum (-1)^{n+1} \frac{n^2}{n^3+1}$ converges.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}$ $\cos(n\pi) = (-1)^n$!

So this is really $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$ $\longrightarrow$ Alternating, with $b_n = \frac{1}{n^{3/4}}$.

So can use Alt. Series Test.