Lecture 37

21 Apr 2010

Last time: Absolute and conditional convergence
Ratio Test

Ex: $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\qquad a_n = \frac{n^n}{n!}$

Ratio Test: look at $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!}$$

[use $(n+1)! = (n+1)n!$]

$$= \frac{(n+1)\cdot(n+1)^n}{n^n} \cdot \frac{n!}{(n+1)n!}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

So $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

Since $e > 1$, Ratio Test says $\sum \frac{n^n}{n!}$ diverges.

Ex $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ $\qquad a_n = \frac{\sqrt{n}}{1+n^2}$

Suppose we try Ratio Test on this:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}\,/\,1+(n+1)^2}{\sqrt{n}\,/\,1+n^2} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}$$

and $\lim\limits_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2} = 1$ (skipped simplification steps here)

So the Ratio Test is inconclusive here.

(Could see that this Σ converges using Limit-Comp Test, with $b_n = \frac{1}{n^{3/2}}$).

Root Test

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$

  Then $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ (or $= \infty$)

  Then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

[If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then the Root Test is inconclusive.]

Ex $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ $\qquad$ $a_n = \left(\frac{2n+3}{3n+2}\right)^n$

Root Test: $\displaystyle\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2}$

and $\displaystyle\lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$

so $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converges (absolutely)

Strategy for Testing Series (Ch 12.7)

Classify the series according to its form.

1) $\sum \frac{1}{n^p}$: p-test.

2) $\sum ar^{n-1}$ or $\sum ar^n$: geometric series: converges if $|r|<1$, diverges if $|r| \geq 1$

3) If the series looks similar to a p-series or geom series: try comparison or limit-comparison (picking b_n to be the p-series or geometric series). (If the series has some negative terms then apply this method instead to $\sum |a_n|$ — i.e. test for absolute convergence.)

4) If you can see easily that $\lim_{n\to\infty} a_n \neq 0$, use Test For Divergence.

5) If the series is of the form $\sum_{1}^{\infty} (-1)^n b_n$ or $\sum_{1}^{\infty} (-1)^{n+1} b_n$

try Alternating Series Test.

6) If the series involves factorials (or other products involving n terms, including k^n) — try Ratio Test.
[But not for series where a_n is just rational function — Ratio Test will be inconclusive for those]

7) If $a_n = (\text{something})^n$ try Root Test.

8) If $a_n = f(n)$ and you know to do $\int_1^{\infty} f(x)\,dx$
[and $f(x)$ is decreasing for large enough x]
try Integral Test.

Ex $\sum_{1}^{\infty} \left(\frac{n^2+4}{3n^2+7n}\right)^{3n}$

$a_n = \left(\frac{n^2+4}{3n^2+7n}\right)^{3n}$ — use Root Test — $\sqrt[n]{\left(\frac{n^2+4}{3n^2+7n}\right)^{3n}}$

$$= \left(\frac{n^2+4}{3n^2+7n}\right)^{3}$$

...

Ex $\sum \frac{n+8}{2n+1}$: use Test For Divergence

Ex $\sum n^2 e^{-n^3}$: use Integral Test
with $f(x) = x^2 e^{-x^3}$

Ex $\sum (-1)^n \frac{n^3}{n^4+1}$: use Alternating Series Test

[and if we want to see whether it's <u>absolutely</u> convergent, use Lim-Comp with $b_n = \frac{1}{n}$]

Ex $\sum \frac{2^k}{k!}$: use Ratio Test

Ex $\sum n \sin\left(\frac{1}{n}\right)$: use Test For Divergence

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Ex $\sum \frac{1}{2+3^n}$: use Comparison or Lim-Comp
with $b_n = \frac{1}{3^n}$

Ex $\sum (-1)^j \frac{\sqrt{j}}{j+5}$: use Alt. Series Test
