Lecture 40

28 Apr 2010

Last time: functions as power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (*)$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad (**)$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$ by applying $\frac{d}{dx}$ to both sides of $(*)$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$ by integrating both sides of $(*)$ with respect to x, and multiplying by -1

Could also similarly get

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$ by integrating both sides of $(**)$

How do we get power series representing a more general function $f(x)$?

Taylor (and Maclaurin) Series (Ch 12.10)

If we have any function f (which is "nice enough" — can be differentiated arbitrarily many times) and any number a, we can write down the Taylor series of f centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

[power series centered at a]

$[0! = 1] \longrightarrow \; = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

If this series has radius of convergence $R > 0$ then its sum is $f(x)$ for $x \in (a-R, a+R)$. i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{for } |x-a| < R$$

If $a = 0$ then we call this series the Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \text{for } |x| < R$$

(So "Maclaurin Series" = "Taylor series centered at $a=0$".)

Ex. Find the Maclaurin Series for $f(x) = e^x$ and its radius of convergence.

Macl. Series: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$
\begin{aligned}
f(x) &= e^x & \qquad f(0) &= e^0 = 1 \\
f^{(1)} = f'(x) &= e^x & f'(0) &= 1 \\
f^{(2)} = f''(x) &= e^x & f''(0) &= 1 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1
\end{aligned}
$$

So the Macl. Series for $f(x) = e^x$ is just $\underline{\underline{\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n}}$

<u>Radius of convergence:</u> use Ratio Test $\qquad a_n = \frac{1}{n!}x^n$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{(n)!}{|x|^n} = |x| \frac{n!}{(n+1)!} = \frac{|x|}{n+1}$$

[using $(n+1)! = (n+1)n!$]

$$\text{So } \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

Since $0<1$, Ratio Test says this series <u>converges</u> — for all values of x. So radius of convergence $\underline{\underline{R = \infty}}$.

$$\left[\text{So } \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \qquad \text{for all } x.\right]$$

<u>Ex</u> Find the Maclaurin series for $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned}
f(x) &= \sin x & f(0) &= \sin(0) = 0 \\
f'(x) &= \cos x & f'(0) &= \cos(0) = 1 \\
f''(x) &= -\sin x & f''(0) &= -\sin(0) = 0 \\
f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -\cos(0) = -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin(0) = 0 \\
&\vdots & &\vdots
\end{aligned}$$

(repeats with period 4) $\qquad$ (repeats with period 4)

Maclaurin series

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 + \frac{f^{(7)}(0)}{7!}x^7 + \cdots$$

$$= 0 + \frac{1}{1!}\cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}}$$

Maclaurin series for $\sin(x)$

What is its radius of convergence?

Like in previous example, apply ratio test: you'll get $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$ for all x.

So $\underline{\underline{\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}}$ for all x.

<u>Ex</u> Find the first 3 terms of the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $a = 9$.

$$f(x) = x^{-1/2} \qquad f(9) = \frac{1}{3}$$

$$f'(x) = -\frac{1}{2}x^{-3/2} \qquad f'(9) = -\frac{1}{54}$$

$$f''(x) = \frac{3}{4}x^{-5/2} \qquad f''(9) = \frac{1}{324}$$

First 3 terms of Taylor series:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \qquad a = 9$$

$$= \frac{1}{3} + \left(-\frac{1}{54}\right)(x-9) + \frac{1}{324\cdot 2}(x-9)^2$$

$$= \frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2$$

[This is also called the "Taylor polynomial of degree 2 centered at $a=9$".]

Ex Find the Maclaurin series for $f(x) = x^3 \sin(x)$.

We could use the general formula for Maclaurin series, but there's a trick: we already know the Maclaurin series for $\sin(x)$.

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

So
$$x^3 \sin(x) = x^3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+1)!} \quad \text{for all } x.$$

Ex Find the Maclaurin series for $\cos(x)$.

$$\cos(x) = \frac{d}{dx} \sin(x).$$
$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-1)^n$$
$$= \sum_{n=0}^{\infty} \frac{(2n+1)\, x^{2n}}{(2n+1)!} (-1)^n$$
$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} (-1)^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$