Lecture 41

30 Apr 2010

Housekeeping:

Exam 3 Tuesday, 7–9pm

- Ch 12.1–12.11
- 18 questions
- About as hard as Exam 2 (~)

Updated notes at http://www.ma.utexas.edu/users/neitzke/408L/
(Including a single PDF containing all lecture notes)

Uses of Taylor/Maclaurin Series (Ch 12.11)

Taylor series for $f(x)$ centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x) \qquad \text{for} \quad |x-a|<R$$

We know a bunch of examples:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x|<1$$

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n} \qquad |x|<1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{all } x$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{all } x$$

Taylor polynomial of f, of degree d, centered at a:

$$T_d(x) = \sum_{n=0}^{d} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

(first few terms of Taylor series: Σ up to d instead of ∞)

Ex Use the Taylor polynomial of degree 2, centered at 0, for e^x to estimate $\sqrt[4]{e}$.

$f(x) = e^x$

Taylor poly: $T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

so $T_2(x) = 1 + x + \frac{x^2}{2}$

To get $\sqrt[4]{e} = e^{1/4}$: just plug in $x = \frac{1}{4}$

$$T_2\left(\tfrac{1}{4}\right) = 1 + \frac{1}{4} + \frac{1}{32} = \underline{\underline{\frac{41}{32}}}$$

$$\left[\begin{array}{ll} \sqrt[4]{e} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots & x = \frac{1}{4} \\ \text{we took } 1 + x + \frac{x^2}{2} & \end{array}\right]$$

Ex Use the Taylor poly. of degree 3 for $\sin(x)$ centered at 0 to estimate $\sin\left(\frac{1}{10}\right)$.

We know Tay. series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The degree 3 Taylor poly. is

$$T_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

Plug in $x = \frac{1}{10}$:

$$T_3\left(\frac{1}{10}\right) = \frac{1}{10} - \frac{\left(\frac{1}{10}\right)^3}{6} = \frac{1}{10} - \frac{1}{6000} = \underline{\underline{\frac{599}{6000}}}$$

Ex Use the Taylor polynomial for e^{-x^2} of degree 2 centered at 0 to estimate

$$\int_{-1/2}^{1/2} e^{-x^2}\,dx.$$

Taylor series for e^{-x^2}:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Replace x by $-x^2$:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$$

To get Taylor poly, keep only terms involving x^d with $d \le 2$:

So $$T_2(x) = 1 - x^2$$

$$\int_{-1/2}^{1/2} T_2(x)\,dx$$

$$= \int_{-1/2}^{1/2} (1-x^2)\,dx = x - \tfrac{1}{3}x^3 \Big|_{-1/2}^{1/2}$$

$$= \underline{\underline{\frac{11}{12}}}$$

Another use of Taylor series:

<u>Ex</u> Calculate $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{\pi^2}{16}\right)^n \frac{1}{(2n)!}$.

[Idea: this looks like the Taylor series for cos(x)...]

$$\sum_{n=0}^{\infty} \left(-\frac{\pi^2}{16}\right)^n \frac{1}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!}$$

But $\displaystyle\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

So $$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

Ex Find $\int_0^t \ln(1+x^3)\,dx$ as a power series.

Remember $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

$$\ln(1+x^3) = \sum_{n=1}^{\infty} -\frac{(-x^3)^n}{n}$$

$\ln(1+x^3) = \ln(1-(-x^3))$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}$$

So $\int_0^t \ln(1+x^3)\,dx = \int_0^t \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}\,dx$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n+1}}{n(3n+1)} \Bigg|_{x=0}^{x=t}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{t^{3n+1}}{n(3n+1)} - \frac{0^{3n+1}}{n(3n+1)} \right]$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{3n+1}}{n(3n+1)}$$

Ex Find the Taylor polynomial of degree 1 for $\sqrt[3]{x}$ centered at $a = 27$.

Taylor polynomial $T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\qquad$ $f(27) = 3$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad$ $f'(27) = \frac{1}{3}(27)^{-2/3} = \frac{1}{3}\left(\frac{1}{9}\right) = \frac{1}{27}$

So $T_1(x) = 3 + \frac{1}{1!} \cdot \frac{1}{27}(x-27)$

$= \underline{\underline{3 + \frac{1}{27}(x-27)}}$