Lecture 4

Admin: HW03 available now, due next Tue 3am

LM usually due M, W, Sat night midnight

but, no LM this W !

Tricky substitutions:

1) $\int \frac{dx}{1+9x^2} = ?$ $\qquad \cancel{\tan^{-1}(3x)} \quad \frac{1}{3}\tan^{-1}(3x)$

Idea: to do this, want to reduce to $\int \frac{du}{1+u^2}$

so, take $u = 3x$, $du = 3\,dx$, $\frac{1}{3}du = dx$

then $\int \frac{dx}{1+9x^2} = \int \frac{\frac{1}{3}du}{1+u^2} = \frac{1}{3}\tan^{-1}(u)$
$= \frac{1}{3}\tan^{-1}(3x)$

(could similarly do eg $\int \frac{1}{\sqrt{1-16x^2}}\,dx$ by $u=4x$)

because we want $u^2 = 16x^2$, $u = 4x$

2) $\int \frac{5}{x^2+6x+10}\,dx$

the trick: "complete the square"

set $u = x+3$ $\qquad du = dx$

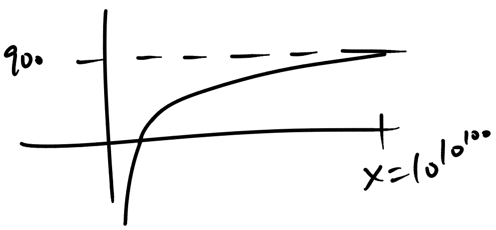
then $u^2 = (x+3)^2 = x^2+6x+9$

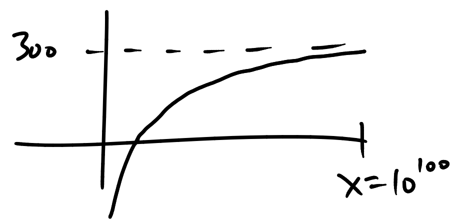
so $\int \frac{5}{x^2+6x+10}\,dx = \int \frac{5}{u^2+1}\,du = 5\tan^{-1}(u) + C$
$= 5\tan^{-1}(x+3) + C$

Q $\int \frac{1}{x\ln x}\,dx = \ln|\ln x| + C \qquad u = \ln x \quad du = \frac{dx}{x}$

$$\int \frac{1}{x \ln x}\,dx = \int \frac{du}{u} = \ln|u| = \underline{\underline{\ln|\ln x| + C}}$$

Q $\lim_{x\to\infty} \ln|\ln x| = \infty$ $\qquad$ $\lim_{x\to\infty} \ln x = \infty$





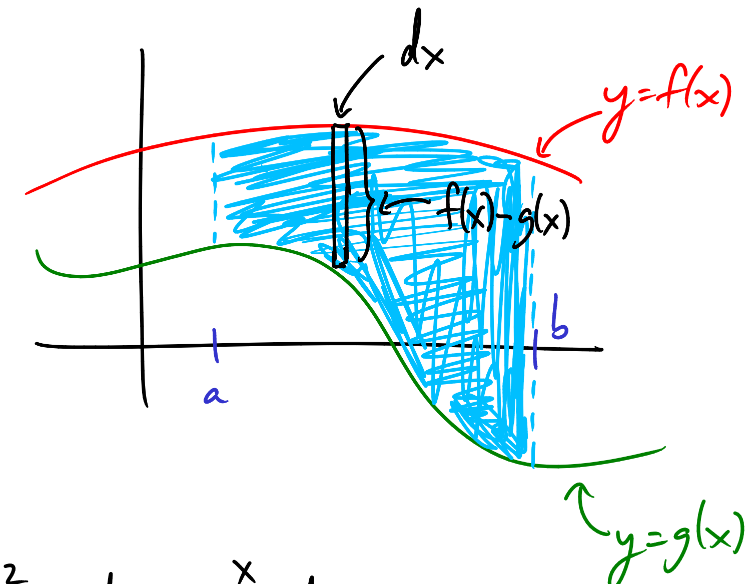
Area between curves

Two curves $y=f(x)$ and $y=g(x)$.

Say $f(x)>g(x)$ for x in $[a,b]$.

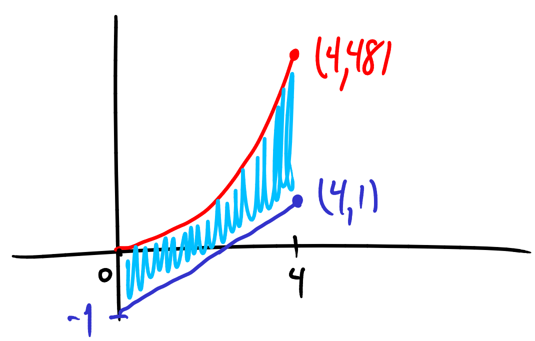
Then the area of the shaded region is

$$\int_a^b \big(f(x)-g(x)\big)\,dx$$



Q Find the area between $y=3x^2$ and $y=\frac{x}{2}-1$ with x between 0 and 4.

64, 67, 56

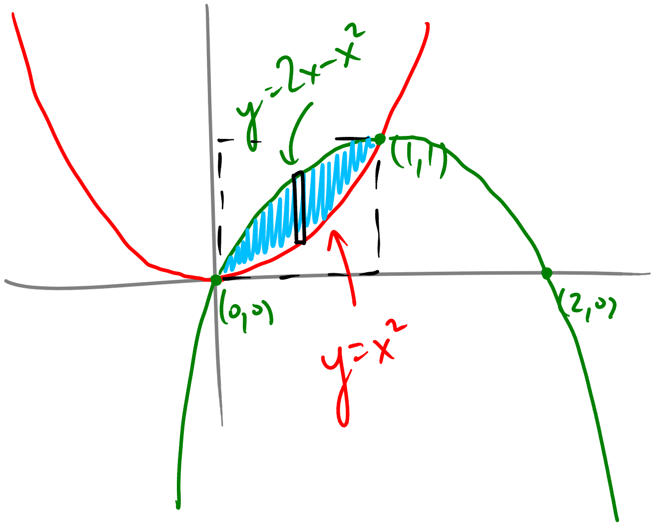


$$A = \int_0^4 \left(3x^2 - \left(\frac{x}{2}-1\right)\right) dx$$

$$= x^3 - \frac{x^2}{4} + x \Big|_0^4 = (64-4+4)-(0+0+0) = \underline{\underline{64}}$$

Q Find the area of the finite region between $y=x^2$ and $y=2x-x^2$.

$A = \frac{1}{3}$



$$A = \int_0^1 (2x-x^2) - (x^2)\, dx$$

$(2x-x^2)$: top, (x^2): bottom

$$= \int_0^1 2x - 2x^2\, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1$$

$$= \frac{1}{3}$$

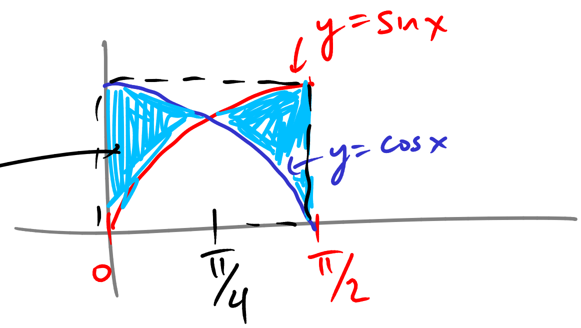
Q Find the area of the region between $y=\sin x$ and $y=\cos x$ for x between 0 and $\pi/2$.

$2\sqrt{2}-2$ $\approx 2.8-2$ ≈ 0.8

$4\sqrt{2}-2$ $\approx 5.6-2$ ≈ 3.6

$2\sqrt{2}$ ≈ 2.8

area of box ≈ 1.5



$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx = \sin x + \cos x \Big|_0^{\pi/4}$$

$$= \left(\sin \tfrac{\pi}{4} + \cos \tfrac{\pi}{4}\right) - (\sin 0 + \cos 0)$$

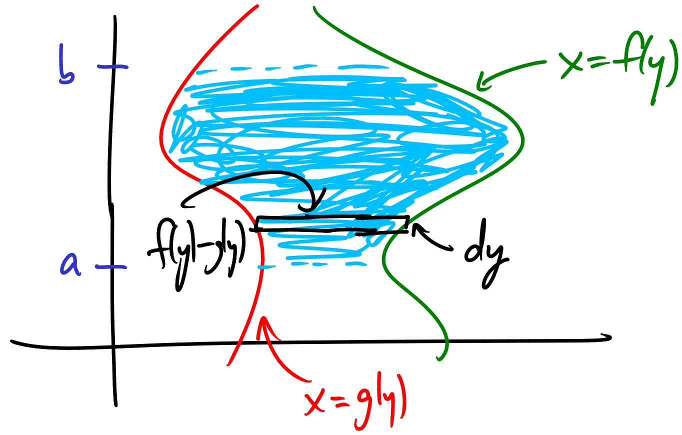
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1)$$

$$= \sqrt{2} - 1$$

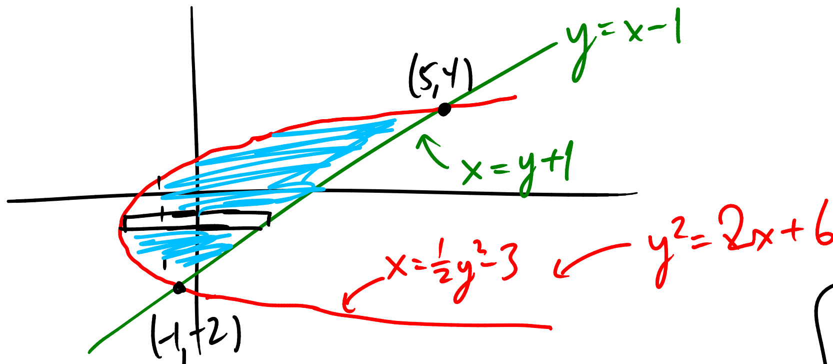
total area $= 2(\sqrt{2}-1) = \underline{2\sqrt{2}-2}$

Can also consider curves: $x = f(y)$
$x = g(y)$

$$A = \int_a^b \left(f(y) - g(y)\right)\, dy$$



Q Find the area between parabola $y^2 = 2x+6$
line $y = x-1$.



to find int pts:
$(x-1)^2 = 2x+6 \ldots$

$$\int_{-2}^{4} (y+1) - \left(\tfrac{1}{2}y^2 - 3\right)\, dy$$

$= \ldots = \underline{\underline{18}}$

Extra Q find area between
$y = x^3 - x^2 - 7x - 4$
$y = -x^2 + 2x - 4$