Lecture 7

Next LM already posted, due Mon night midnight

My office hr today 5-6pm RLM 9.134

(+M 2-3pm)

Exam 1 Tue Oct 3 7-9pm covers Sec 5.3-7.1

Jester A121A (from start to int. by parts)

review probably Thu before

Trigonometric integrals

Q $\int \sin^5\theta \underbrace{\cos\theta\, d\theta}_{du} = ?$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int u^5\, du$

$= \frac{1}{6}u^6 + C$

$= \frac{1}{6}\sin^6\theta + C$

Similarly for $\int \sin^a\theta \cos\theta\, d\theta$ (use $u = \sin\theta$)

or $\int \cos^a\theta \sin\theta\, d\theta$ (use $u = \cos\theta$)

Q $\int \sin^3\theta\, d\theta = ?$

use $\sin^2\theta + \cos^2\theta = 1$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$= \int \sin\theta(1-\cos^2\theta)\, d\theta$

$= -\int (1-u^2)\, du$

$= -u + \frac{u^3}{3} = \boxed{-\cos\theta + \frac{\cos^3\theta}{3} + C}$

Q $\int \sin^5\theta \cos^2\theta \, d\theta = ?$

either reduce to $\int \sin^a\theta \cos\theta \, d\theta$

or $\int \cos^a\theta \sin\theta \, d\theta$

$= \int \sin^4\theta \cos^2\theta \, (\sin\theta \, d\theta)$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$

$= \int (\sin^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$

$= \int (1-u^2)^2 \, u^2 \quad (-du)$

$= -\int (1-2u^2+u^4) \, u^2 \, du$

$= -\int u^2 - 2u^4 + u^6 \, du$

$= -\left(\frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7}\right) + C$

$= -\frac{\cos^3\theta}{3} + \frac{2}{5}\cos^5\theta - \frac{1}{7}\cos^7\theta + C$

General rule for $\int \sin^a\theta \cos^b\theta \, d\theta$:

- If a is odd, pick off one $\sin\theta \, d\theta$
  then use $\sin^2\theta = 1-\cos^2\theta$ to rewrite remaining sines in terms of $\cos\theta$
  then use $u = \cos\theta$
- If b is odd, pick off one $\cos\theta \, d\theta$
  then use $\cos^2\theta = 1-\sin^2\theta$ to write remaining cosines in terms of $\sin\theta$
  then use $u = \sin\theta$

What about even powers?

Q $\int \sin^2\theta \, d\theta$

and $\int_0^{\pi} \sin^2\theta \, d\theta$

Half-angle formulas:
$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\int \sin^2\theta \, d\theta = \cancel{\int \sin\theta (\sin\theta \, d\theta)} = \cancel{\int (1-\cos^2\theta)\, d\theta}$$

$$= \int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C$$

$$\int_0^{\pi} \sin^2\theta \, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_0^{\pi} = \left(\frac{\pi}{2} - 0\right) - (0-0) = \frac{\pi}{2}$$

"the average value of $\sin^2\theta$ is $\frac{1}{2}$"

$$\cos^2\theta + \sin^2\theta = 1$$

Q $\int \cos^4\theta \, d\theta = ?$

$$= \int (\cos^2\theta)^2 \, d\theta$$

$$= \int \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

$$= \frac{1}{4}\int 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$
$$\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$$

$$= \frac{1}{4}\left(\int 1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\, d\theta\right)$$

$$= \cdots = \frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C$$

Q $\int \tan^6 x \sec^4 x \, dx = ?$

$$\left[\begin{array}{ll} u = \tan x & \sec^2 x = 1 + \tan^2 x \\ du = \sec^2 x \, dx & \end{array}\right]$$

$$\int \tan^6 x \sec^2 x \, (\sec^2 x \, dx)$$

$$= \int \tan^6 x \,(1+\tan^2 x)\, \sec^2 x \, dx$$

$$= \int u^6 (1+u^2) \, du$$

$$\left(\begin{array}{c} \frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2} \\ \tan^2 + 1 = \sec^2 \end{array} \right)$$

$= \int (u^6+u^8)\,du = \frac{1}{7}u^7+\frac{1}{9}u^9+C = \underline{\frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C}$

Same strategy for $\int \tan^a x \sec^b x$ when b is <u>even</u>

<u>Q</u> $\int \tan^3 x \sec^5 x \, dx$

$$\begin{bmatrix} u = \sec x \\ du = \sec x \tan x \, dx \end{bmatrix}$$

$= \int \tan^2 x \sec^4 x \,(\sec x \tan x \, dx)$

$= \int (\sec^2 x - 1) \sec^4 x \,(\sec x \tan x \, dx)$

$= \int (u^2-1)\, u^4 \, du = \int u^6 - u^4 \, du = \frac{u^7}{7} - \frac{u^5}{5} + C = \underline{\frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C}$

Same strategy for $\int \tan^a x \sec^b x \, dx$ when a is <u>odd</u>, $b \geq 1$

Handy facts: $\int \tan x \, dx = \ln|\sec x| + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

<u>Ex</u> $\int \tan^3 x \, dx = \int \tan x (\sec^2 x - 1)\, dx$

$= \int (\tan x \sec^2 x - \tan x)\, dx$

$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u \, du - \int \tan x \, dx$

$= \frac{1}{2}u^2 - \int \tan x \, dx$

$= \underline{\frac{1}{2}\tan^2 x - \ln|\sec x| + C}$

Ex $\int \sin 4x \cos 7x \, dx$

Use identities:

$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$$= \frac{1}{2}\int \sin(-3x) + \sin(11x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{3}\cos 3x - \frac{1}{11}\cos 11x\right) + C$$