Lecture 8

Exam 1 next Tue 7–9pm Jester A121A

covers material from beginning of term thru int by parts

bring: pencils, ID

no calculators

in-class exam review Thu

HW05 due Tue 3am

Last time: trigonometric integrals — like $\int \sin^a\theta \cos^b\theta \, d\theta$ or $\int \sec^a\theta \tan^b\theta \, d\theta$

strategy: u-sub

$u = \sin\theta \quad du = \cos\theta \, d\theta$

or $u = \cos\theta \quad du = -\sin\theta \, d\theta$

$u = \sec\theta \quad du = \sec\theta\tan\theta \, d\theta$

$u = \tan\theta \quad du = \sec^2\theta \, d\theta$

One more example:

$$\int \sec^3 x \, dx$$

could try: $\int \sec x \, (\sec^2 x \, dx) \qquad u = \tan x$

$\int \sec x \quad du$

— difficult

IBP: $u = \sec x \quad v = \tan x$

$du = \sec x \tan x \, dx \quad dv = \sec^2 x \, dx$

$$\begin{aligned}
\int \sec^3 x \, dx &= \sec x \tan x - \int \tan x \, (\sec x \tan x \, dx) \\
&= \sec x \tan x - \int \sec x \tan^2 x \, dx \\
&= \sec x \tan x - \int \sec x \, (\sec^2 x - 1) \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx
\end{aligned}$$

which trig id. to memorize?

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx$

So $2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$

$$\int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

Q what about $\int \sec^4 x \, dx$?

$\int \sec^2 x \cdot (\sec^2 x \, dx)$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int \sec^2 x \cdot du = \int (\tan^2 x + 1) \, du$

$= \int (u^2+1) \, du$

$= \dots$

Trigonometric Substitution

The idea: to deal with integrals involving $\sqrt{}$ like $\sqrt{1-x^2}$

want $1-x^2$ to be $(\text{something})^2$.

So, let $x = \sin\theta$; then $1-x^2 = 1-\sin^2\theta = \cos^2\theta$

so $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

Q $\int \frac{\sqrt{9-x^2}}{x^2} \, dx = ?$

$x = 3\sin\theta$

$1 - \sin^2\theta = \cos^2\theta$

$\underbrace{9 - 9\sin^2\theta}_{x^2} = 9\cos^2\theta$

$x^2 \rightarrow x = 3\sin\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta \, d\theta$

$$\int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta \, d\theta$$

$$= \int \frac{\sqrt{9\cos^2\theta}}{9\sin^2\theta} 3\cos\theta \, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta} \cos\theta \, d\theta$$

$\sin^2 + \cos^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$

$$= \int \cot^2\theta \, d\theta$$

$\cot^2\theta = \csc^2\theta - 1$

$$= \int (\csc^2\theta - 1) \, d\theta$$

recall $\int \sec^2\theta \, d\theta = \tan\theta + C$

$$= -\cot\theta - \theta$$

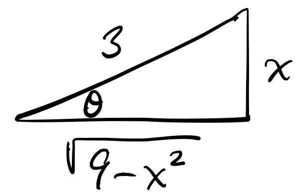
$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

$\sin^{-1}\left(\frac{x}{3}\right) = \theta$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$ $= \frac{\text{opp}}{\text{hyp}}$



$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{9-x^2}}{x}$

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right)$$

Q What is $\int \frac{1}{x^2\sqrt{x^2+4}} dx$?

need a identity like

$(\text{something})^2 + (\text{something})^2 = (\text{something})^2$

$\tan^2 + 1 = \sec^2$

$4\tan^2\theta + 4 = 4\sec^2\theta$

so want $x^2 = 4\tan^2\theta$

$x = 2\tan\theta$

take $x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$$\int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} 2\sec^2\theta \, d\theta$$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4\sec^2\theta}}\, 2\sec^2\theta \, d\theta$$

$$= \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta} \cdot 2\sec^2\theta \, d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta \qquad u = \sin\theta, \quad du = \cos\theta \, d\theta$$

$$\left(\text{or: } \frac{1}{4}\int \cot\theta \csc\theta \, d\theta = \frac{1}{4}(-\csc\theta) \right)$$

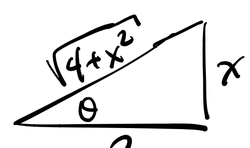
$$= \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) = -\frac{1}{4\sin\theta}$$

$$x = 2\tan\theta$$

$$\frac{x}{2} = \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$= -\frac{1}{4}\csc\theta$$



$$= -\frac{1}{4}\frac{\sqrt{4+x^2}}{x} + C$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{4+x^2}}{x}$$

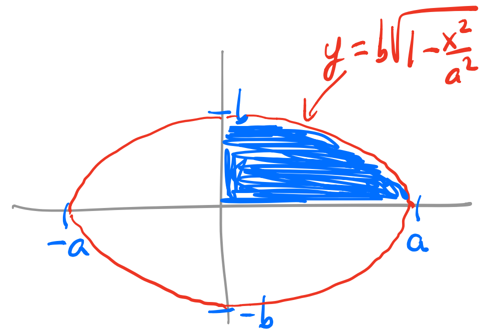
Table: $\sqrt{a^2-x^2}$ use $x = a\sin\theta$, $1-\sin^2 = \cos^2$

$\sqrt{a^2+x^2}$ use $x = a\tan\theta$ $\quad 1+\tan^2 = \sec^2$

$\sqrt{x^2-a^2}$ use $x = a\sec\theta$ $\quad \sec^2 - 1 = \tan^2$

Q. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right) \qquad y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx \qquad ab(\pi+1)$$

$$=4b\int_0^a\sqrt{1-\frac{x^2}{a^2}}\,dx \qquad 1-\sin^2\theta=\cos^2\theta$$

$$\frac{x^2}{a^2}=\sin^2\theta$$

$$x^2=a^2\sin^2\theta$$

$$x=a\sin\theta$$

$x=a\sin\theta$

$dx=a\cos\theta\,d\theta$

$$=4b\int_{x=0}^{x=a}\sqrt{1-\frac{a^2\sin^2\theta}{a^2}}\;a\cos\theta\,d\theta$$

$$=4ab\int_{x=0}^{x=a}\sqrt{1-\sin^2\theta}\;\cos\theta\,d\theta$$

$x=a\sin\theta$

$x=0 \longrightarrow \sin\theta=0 \longrightarrow \theta=0$

$x=a \longrightarrow a\sin\theta=a$

$\sin\theta=1 \longrightarrow \theta=\pi/2$

$$=4ab\int_{x=0}^{x=a}\cos\theta\cos\theta\,d\theta$$

$$=4ab\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=4ab\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$\Big|^{\pi/2}$

$$= 4ab \cdot \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0$$

$$= 4ab \cdot \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \underline{\underline{\pi \cdot a \cdot b}}$$